# DSP: Z-Transform and Basic of Filters

| Created by | Ⓑ Borhan |
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| Tag | Year 3 Term 1 |

**Z Transform**

- Discrete time LTI system z transform is used.
- Z transform is mathematical tool used for conversion of time domain into frequency domain (z domain) and is a function of the complex valued variable Z.
- Biliteral/Two sided

  $X(z) = Z(x(n)) = \sum_{n=-\infty}^{n=\infty} x(n)Z^{-n}$
- Uniliteral/One sided

  $X(z) = Z(x(n)) = \sum_{n=0}^{n=\infty} x(n)Z^{-n}$
- if $x[n] = 0$ for $n < 0$, then Biliteral = Uniliteral

**Geometric Series Sum formula**

- Infinite
  - $\sum_0^\infty c^n = \frac{1}{1-c}$
  - $\sum_k^\infty c^n = \frac{c^k}{1-c}$

  $0 < |c| < 1$
- Finite
  - $c \neq 1$

    $\sum_0^{N-1} c^n = \frac{c^N-1}{c-1}, \sum_0^N c^n = \frac{c^{N+1}-1}{c-1}$
  - $c = 1$

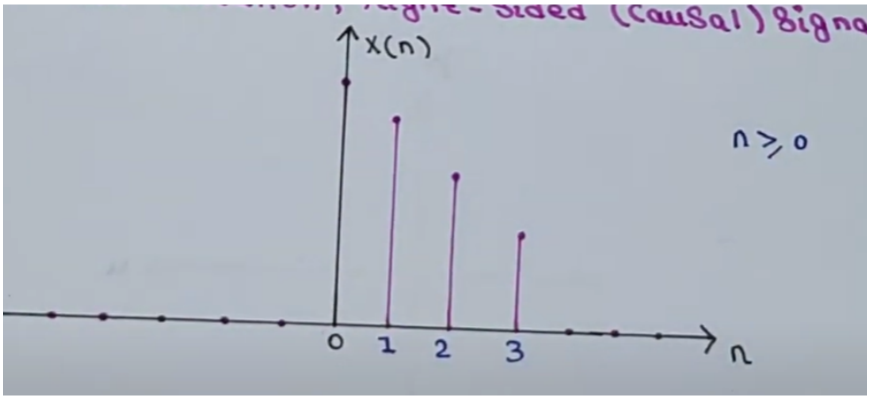
$$\sum_{0}^{N-1} c^n = N, \sum_{0}^{N} c^n = N+1$$

($c$ is a complex constant)

**ROC**

- The set of $Z$ for which $X(z)$ is converges (gives finite value)/the set of points in $Z$-plane for which $X(z)$ is converges is called *Region Of Converges.*
- If there's no value of $z$ where $X(z)$ is converges, then sequence of $x(n)$ is said to be having no $Z$ transform.

- **ROC for finite duration signal, right-sided (causal) signal**



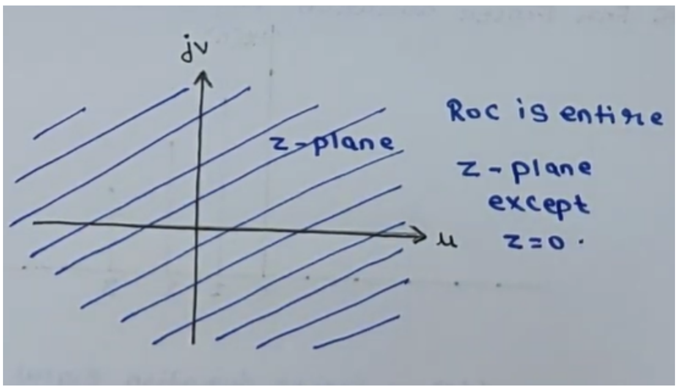

N samples, $0 \le n \le N-1$

$x(n) = \{x(0), x(1), ...x(n)\}$

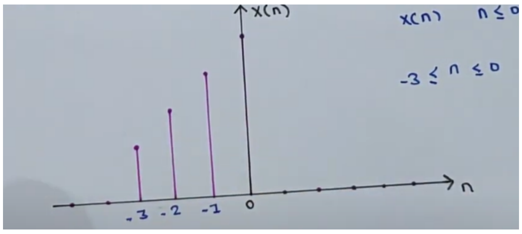
$X(z) = \sum_{n=0}^{n=\infty} x(n)z^{-n} = x(0) + x(1)z^{-1} + ...x(n-1)z^{-(n-1)}$

if $z = 0$, $X(z)$ is infinite.

$X(z)$ exists for all values of z, except $z = 0$



- **ROC for finite duration, left sided (anti causal)**

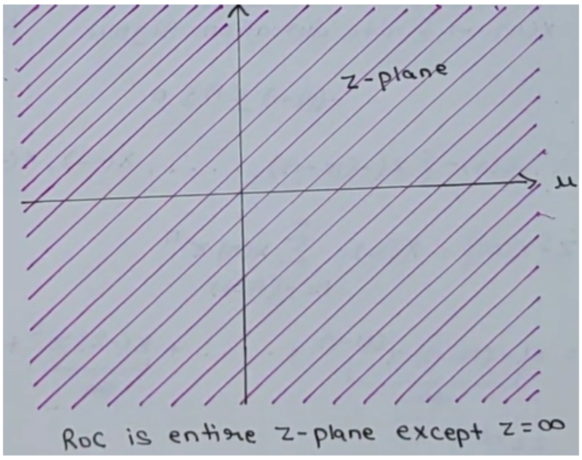



Defined : $-(N-1) \leq n \leq 0$

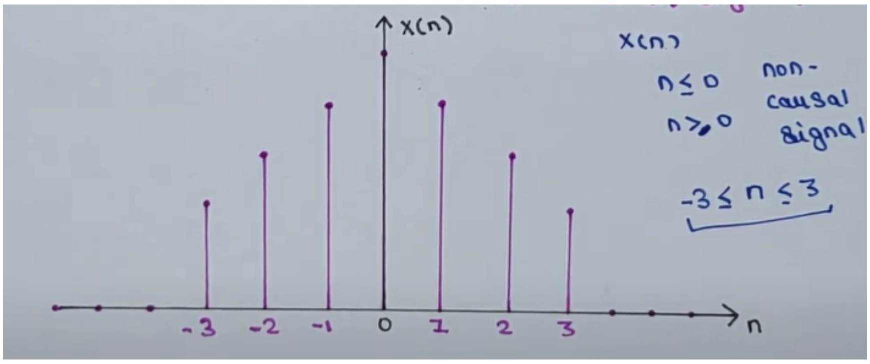
$x(n) = \{x(-(n-1)...x(0)\}$

$X(z) = \sum_{n=-(N-1)}^{0} x(n)z^{-n} = x(-(N-1))z^{N-1} + ... + x(0)$

if $z = \infty$, $X(z)$ is infinite. $X(z)$ exists for all values of z, except $z = \infty$.




- **ROC for finite duration, two sided (non causal signal)**

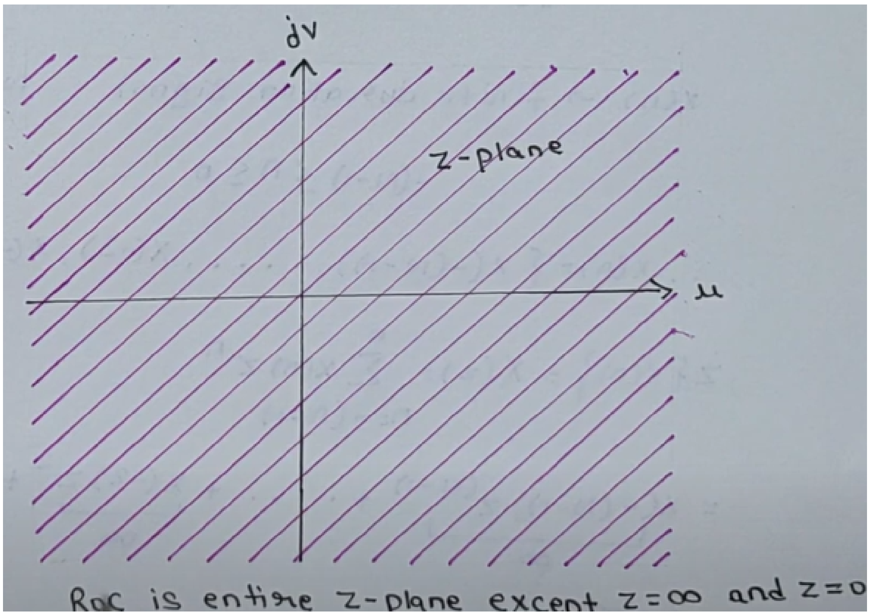



$-m \leq n \leq +m$, $m = \frac{N-1}{2}$, N samples

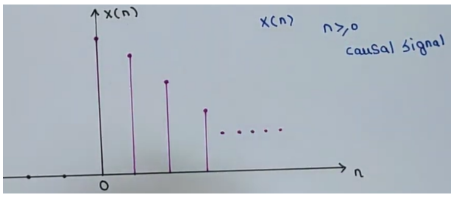
$x(n) = \{x(-m), ..x(0)...x(m)\}$

$X(z) = \sum_{-m}^{+m} x(n) z^{-n} = x(-m)z^{m} + ...x(0)... + x(m)z^{-m}$

$z = 0, \infty$, there are no finite value. ROC = entire $Z - plane$ except $Z = 0, \infty$




- **ROC for infinite duration, right-sided (causal) signal**




Let, $x(n) = r^{n}; n > 0$

$X(z) = \sum_{n=0}^{\infty} (rz^{-1})^{n}$

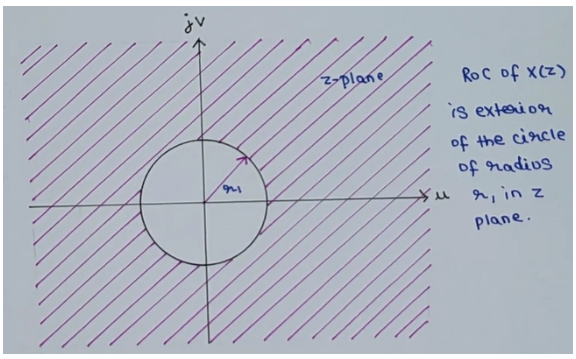
if $0 < |rz^{-1}| < 1,$

$X(z) = \frac{1}{1 - rz^{-1}}$

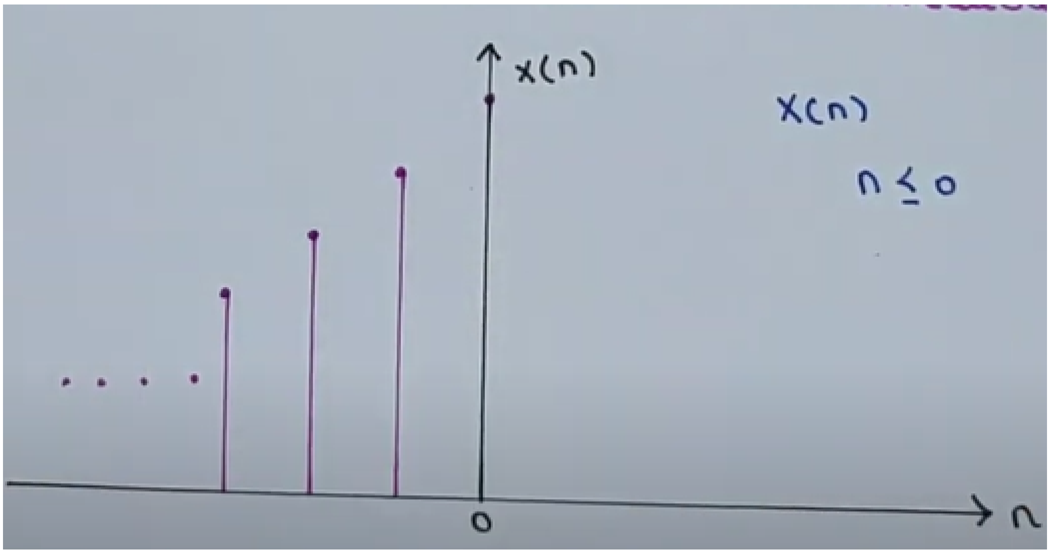
The condition to be satisfied for the convergence of $X(z)$

$0 < |rz^{-1}| < 1$

$|rz^{-1}| < 1 \Rightarrow |z| > |r|$



- **ROC for infinite duration, left-sided(anti causal) signal**




$X(z) = \sum_{n=-\infty}^{0} (r_2 z^{-1})^n = \sum_{n=0}^{\infty} (r_2^{-1} z)^n$

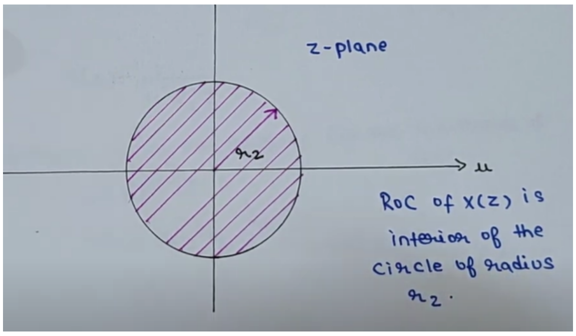
if, $0 < |r_2^{-1} z| < 1$

$X(z) = \frac{1}{1 - r_2^{-1} z}$

Condition to be satisfied for the convergence,

$|r_2^{-1} z| < 1$

⇒ $|z| < |r_2|$




- **ROC for infinite duration, two-sided (non causal) signal**

Let, $x(n) = r_1^n u(n) + r_2^n y(-n), -\infty \leq n \leq +\infty$

$X(z) = \sum_{n=-\infty}^{\infty} [r_1^n u(n) + r_2^n y(-n)] z^{-n}$

$= \sum_{n=-\infty}^{0} r_2^n z^{-n} + \sum_{n=0}^{\infty} r_1^n z^n$

$= \sum_{n=0}^{\infty} r_2^{-n} z^n + \sum_{n=0}^{\infty} r_1^n z^{-n}$

$= \sum_{n=0}^{\infty} (r_2^{-1} z)^n + \sum_{n=0}^{\infty} (r_1 z^{-1})^n$

If $0 < |r_2^{-1} z| < 1$ and $0 < |r_1 z^{-1}| < 1$ then

$X(z) = \frac{1}{1 - r_2^{-1} z} + \frac{1}{1 - r_1 z^{-1}}$

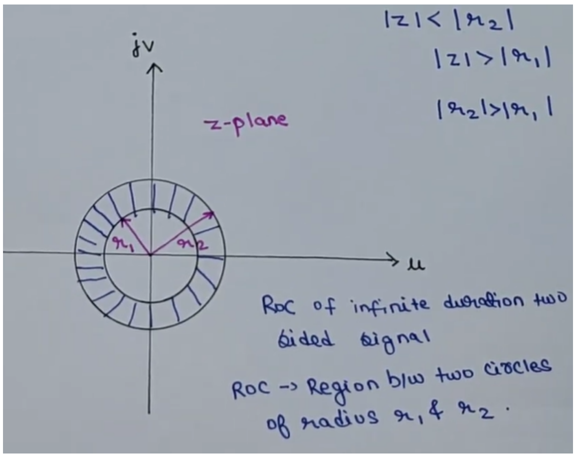
Converges if,

$|r_1 z^{-1}| < 1$

⇒ $|r_1| < |z|$

$|r_2^{-1} z| < 1$

⇒ $|z| < |r_2|$




**Properties of $Z$ Transform**

- **Linearity Property :** The linearity property of Z-transform states that the Z-transform of a weighted sum of two discrete time signals is equal to the weighted sum of individual Z-transform of that system.

$a_1 x_1(n) + a_2 x_2(n) \leftrightarrow a_1 X(z) + a_2 X(z)$

- **Shifting Property :** shifting of $m$-units obtained by multiplying $z^m$

$x(n) \leftrightarrow X(z), -\infty < n < +\infty$

$x(n-m) \leftrightarrow X(z) z^{-m}$

$x(n+m) \leftrightarrow X(z) z^m$

- **Shifting Property of One-sided**

$x(n) \leftrightarrow X(z), 0 < n < +\infty$

$x(n-m) \leftrightarrow z^{-m}X(z) + \sum_{i=1}^{m} x(-i)z^{-(m-i)}$

$x(n+m) \leftrightarrow z^{m}X(z) - \sum_{i=1}^{m} x(i)z^{(m-i)}$

- **Multiplication property**

$x(n) \leftrightarrow X(z)$, then

$nx(n) \leftrightarrow -z\frac{d}{dx}X(z)$

- **Multiplication by an exponential sequence property**

$x(n) \leftrightarrow X(z)$, then

$a^n x(n) \leftrightarrow X(\frac{z}{a})$

- **Time Reversal Property**

$x(n) \leftrightarrow X(z)$, then

$x(-n) \leftrightarrow X(z^{-1})$

- **Conjugation Property**

$x(n) \leftrightarrow X(z)$, then

$x^*(n) \leftrightarrow X^*(z^*)$

- **Convolution Property**

$x_1(n) \leftrightarrow X_1(z), x_2(n) \leftrightarrow X_2(z)$

$x_1(n) * x_2(n) \leftrightarrow X_1(z)X_2(z)$

$[\text{General convulotion} : x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m)]$

- **Correlation property**

$x(n) \leftrightarrow X(z), y(n) \leftrightarrow Y(z)$ then

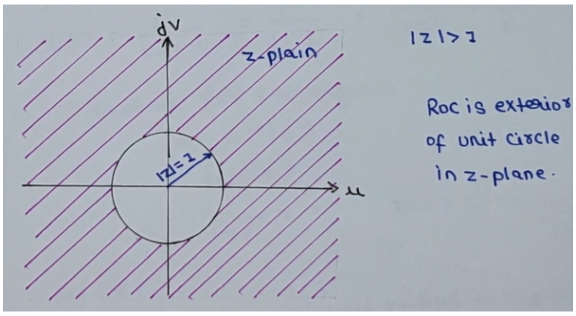
Z transform of $r_{xy}(m) \leftrightarrow X(z)Y(z^{-1})$

$[r_{xy} = \sum_{m=-\infty}^{\infty} x(n)y(n-m)]$

**Finding Z transform**

- $x(n) = u(n)$

$$x(n) = \begin{cases} 1; n \geq 0 \\ 0; n < 0 \end{cases}$$




$X(z) = \sum_{n=0}^{\infty} \mu(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n}$

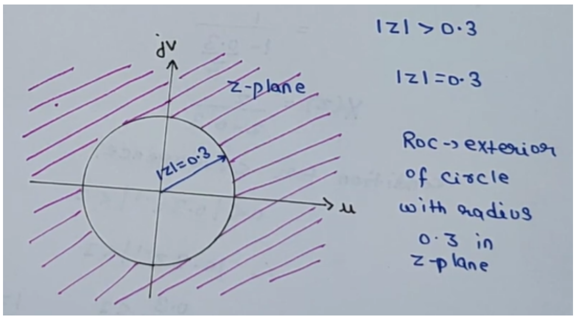
$0 < |z^{-1}| < 1,$

$X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

Condition for converges, $|z^{-1}| < 1$ ⇒ $|z| > 1,$

- $x(n) = (0.3)^n u(n)$

$$x(n) = \begin{cases} 0.3^n; n \geq 0 \\ 0; n < 0 \end{cases}$$




$X(z) = \sum_{n=0}^{\infty} 0.3^n z^{-n}$
$= \sum_{n=0}^{\infty} (0.3z^{-1})^n$

if $0 < |0.3z^{-1}| < 1,$

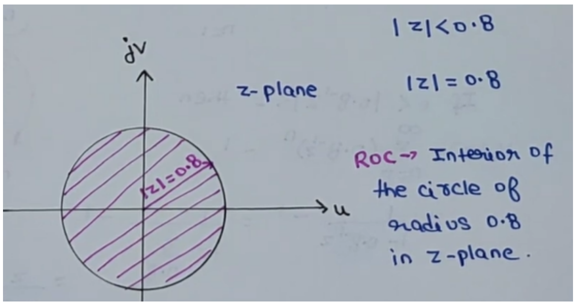
$X(z) = \frac{1}{1-(0.3z^{-1})} = \frac{z}{z-0.3}$

Condition for converges. $|0.3z^{-1}| < 1$ ⇒ $|z| > 0.3$

- $x(n) = (0.8)^n u(-n-1)$

$$u(n) = \begin{cases} 1; -n-1 \geq 0 \\ 0; n \leq 0 \end{cases}$$

$$x(n) = \begin{cases} 0.8^n; n \leq -1 \\ 0; n \leq 0 \end{cases}$$




$X(z) = \sum_{n=-\infty}^{-1} 0.8^n z^{-n} =$
$\sum_{n=1}^{\infty} 0.8^{-n} z^n =$
$\sum_{n=1}^{\infty} (0.8^{-1} z)^n = \frac{(0.8^{-1} z)}{1-0.8^{-1} z}$

If $0 < |0.8^{-1} z| < 1$, then

$X(z) = \frac{0.8^{-1} z}{1-0.8^{-1} z} = \frac{z}{0.8-z}$

Condition for converges, $|0.8^{-1} z| < 1$ ⇒ $|z| < 0.8$

**Poles and Zeroes of Rational Function of $Z$**

$X(z) = \frac{N(z)}{D(z)}$, if $X(z)$ expressed as a ratio of two polynomials $z \text{ or } z^{-1}$, then $X(z)$ is called rational function of $z$.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + .... + b_m z^{-m}}{a_0 + a_1 z^{-1} + .... + a_n z^{-n}} = \frac{b_0}{a_0} \frac{1 + \frac{b_1}{b_0} z^{-1} + ... + \frac{b_m}{b_0} z^{-m}}{1 + \frac{a_1}{a_0} z^{-1} + ... + \frac{a_m}{a_0} z^{-n}} \quad .... (i)$$

$$= G \frac{z^{-m}(z^m + \frac{b_1}{b_0} z^{m-1} + ... + \frac{b_m}{b_0})}{z^{-n}(z^n + \frac{a_1}{a_0} z^{n-1} + ... + \frac{a_n}{a_0})}, [\text{Scaling Factor, G} = \frac{b_0}{a_0}]$$

$$= G \frac{(z-z_1)(z-z_2)...(z-z_n)}{(z-p_1)(z-z_2)...(z-p_n)} \ [n = m]$$

$z_1, z_2, ..., z_n$ : roots of numerator polynomial, zeros of $X(z)$, marked by ○

$p_1, p_2, .., p_n$ : roots of  denominator polynomial, poles of $X(z)$, marked by $X$

- **Inverse Z Transform :**

- $Z^{-1}(x(z)) = x(n) = \frac{1}{2\pi \hat{j}} \oint_c X(z) z^{n-1} dz$

**Questions:**

| | | | |
|---|---|---|---|
| 1. | a) | Write the advantages of Z-transform system over DTFT system. Show the transfer function in Z-transform with equations. Distinguish between zeros and poles in Z-plane. | 8 |
| | b) | If $(Z) = \frac{Z^2+Z+1}{Z^2+0.5Z+0.75}$ , Find out: a). its transfer function representation, b). its difference equation representation. | 4 |
| | c) | If X(z) = $\frac{2+Z+3Z^2}{5+9Z+0.5Z^3}$ , find the values of input and output coefficients. | 2 |

**(a)**

***Advantages of Z-transform over DTFT system:***

- $\delta(n), u(n)$ can't be analyzed by DTFT but Z.
- The transient response of a system **due to initial condition** or due to changing inputs cannot be computed by using DTFT but Z.
- The Z-transform might **exist anywhere in the Z-plane**; the DTFT can only exist on the **unit circle**.
- Z transform allows for the discrete time system **analysis in the frequency domain**. This facilitates the analysis of system characteristics such as **stability, causality, linearity and time and frequency** response.
- Z transform allows the representations of signals and systems in the **frequency domain**.
- It becomes possible to analyze frequency content of signals **determine spectral properties, apply filtering and modulation.**
- Useful for **determining the stability** of DT system.
- It **has convolution property** that simplifies the analysis of linear time invariant system.
- **Simplifies mathematical** operation.

***The Transfer Function***

The transfer function $H(z)$ of a DT linear system is defined as the ration of the Z-transform of the output signal $Y(z)$ to the Z-transform of the input signal $X(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{n=-\infty}^{+-\infty} b_n z^{-n}}{\sum_{m=-\infty}^{+-\infty} a_m z^{-m}}$

$= \frac{b_0+b_1z^{-1}+...+b_mz^{-m}}{a_0+a_1z^{-1}+...+a_nz^{-n}}$

Where, $b_0, b_1, ...$ are input coefficient and $a_0, a_1 ...$ are output.

if $a_o = 1$, then this called the standard form of transfer function.

If we can convert, $H(z) = G\frac{(z-z_1)(z-z_2)...(z-z_n)}{(z-p_1)(z-p_2)...(z-p_n)}$

Then, $z_1, z_2 ...$ are called zeros and $p_1, p_2 ....$ are called poles.

***Distinguish between Zeros and Poles***

| Poles | Zeros |
|---|---|
| A pole of the transfer function H(z) is value of z that makes the denominator zero. | A zero of the transfer function H(z) is a value of $z$ that makes the numerator zero. |
| Mathematical : H(z) = N(z)/D(z)<br>If D(z) = 0, at z=zp, then<br>H(z) → infinity as z→ zp | Mathematical : H(z) = N(z)/D(z)<br>If N(z) = 0, at z=zs, then<br>H(z) → 0 as z→ zs |
| Influence system stability and response | Influence frequency response |
| Poles outside the unit circle indicate instability | Zeros do not affect stability directly |
| Represented by "X" in the Z-plane | Represented by "O" in the Z-plane |

**(b)** *Transfer function representation :* $X(z) = \frac{N(z)}{D(z)} = \frac{b_0+b_1z^{-1}+....+b_mz^{-m}}{1+a_1z^{-1}+....+a_nz^{-n}} = \frac{b_0+b_1z^{-1}+....+b_mz^{-m}}{a_0+a_1z^{-1}+....+a_nz^{-n}}$ *[$a_0 = 1$ means normalized] .$z = e^{j\omega}$*

$H(z) = \frac{1+z^{-1}+z^{-2}}{1+0.5z^{-1}+0.75z^{-2}}$ [Dividing by $z^2$]

$H(e^{j\omega}) = \frac{e^{2j\omega}+e^{2\omega}+1}{e^{2j\omega}+0.5e^{j\omega}+0.75}$ [Replace $z = e^{j\omega}$ on given $X(z)$ of the question]

*Difference equation representation : Transfer function → Inverse Z-transform* → $y(n)$

$H(z) = \frac{1+z^{-1}+z^{-2}}{1+0.5z^{-1}+0.75z^{-2}}$, [Transfer function, $H(z) = \frac{Y(z)}{X(z)} = ..$ ]

$X(z) + z^{-1}X(z) + z^{-2}X(z) = Y(z) + 0.5z^{-1}Y(z) + 0.75z^{-2}Y(z)$ *[cross product]*

⇒ $x[n] + x[n-1] + x[n-2] = y[n] + 0.5y[n-1] + 0.75y[n-2]$ *[Inverse Z-function]*

⇒ $y[n] = x[n] + x[n-1] + x[n-2] + 0.5y[n-1] + 0.75y[n-2]]$

**(c)**

*Idea :* $X(z) = \frac{N(z)}{D(z)} = \frac{b_0+b_1z^{-1}+....+b_mz^{-m}}{a_0+a_1z^{-1}+....+a_nz^{-n}}$*, make given* $X(z)$ *this format and compare with this (* $a_0 = 1$ *) where* $b_0, ...., b_n$ *= input and* $a_0 + ...a_n =$ *output coefficient.*

$X(z) = \frac{2+z+3z^2}{0.5z^3(10z^{-3}+18z^{-2}+1)} = \frac{4z^{-3}+2z^{-2}+6z^{-1}}{10z^{-3}+18z^{-2}+1}$ [Normalized]

Input coefficient : 4, 2, 6

Output Coefficient : 1, 0, 18, 10

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Show the transfer function in Z-transform with equations. Compare zeros and poles in Z-plane.

If $H(Z) = \frac{Z+1}{Z^2 - 0.9Z + 0.81}$, Find out: a). its transfer function representation, b). its difference equation representation.

If $X(z) = \frac{2+z^2+3z^4}{3+4z+z^2}$, find the values of input and output coefficients.

**(a)** Check the answer above.

**(b)** *Transfer function representation:*

$H(z) = \frac{z^{-2}+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$

$H(z) = \frac{e^{j\omega+1}}{e^{2j\omega}+0.9e^{j\omega}+0/81}$

*Difference equation representation*

$H(z) = \frac{z^{-2}+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}} = \frac{Y(z)}{X(z)}$

⇒ $z^{-2}X(z) + z^{-1}X(z) = Y(z) + 0.9z^{-1}Y(z) + 0.82z^{-2}Y(z)$

⇒ $x[n-2] + x[n-1] = y[n] + 0.9y[n-1] + 0.81y[n-2]$

**(c)** $X(z) = \frac{3+z^{-2}+2z^{-4}}{z^{-2}+4z^{-3}+3z^{-4}}$

**Input Coefficients: 3, 0, 1, 0,2**

**Output Coefficients: 0, 0, 1, 4, 3**

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- **Given difference equation, find transfer function representation.**

$y[n] = x[n] + x[n-1] + x[n-2] + 0.5y[n-1] + 0.75y[n-2]]$

⇒ $Y(z) = X(z) + z^{-1}X(z) + z^{-2}X(z) + 0.5z^{-1}Y(z) + 0.75z^{-2}Y(z)$

⇒ $\frac{Y(z)}{X(z)} = ....$

⇒ $H(z) = \frac{Y(z)}{X(z)} = ....$

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2. a) What is meant by region of convergence? Write the properties of ROC. 3

b) Determine the Z-transform and ROC of the signal $x(n) = \left(-\frac{1}{2}\right)^n u[-n-1]$ where $n \leq -1$. 4

c) Shortly describe unite circle in z-domain. A causal LTI system has impulse response h[n] for which the z-transform is $H(z) = \frac{1+z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$, find the region of convergence of $H(z)$. 1+2

**(a)** The set of $Z$ for which $X(z)$ is converges (gives finite value)/the set of points in $Z$-plane for which $X(z)$ is converges is called *Region Of Converges.*

**Properties:**

## Properties of ROC :

(1) For a finite duration right-sided signal (causal signal), the ROC will be entire z-plane except $z=0$

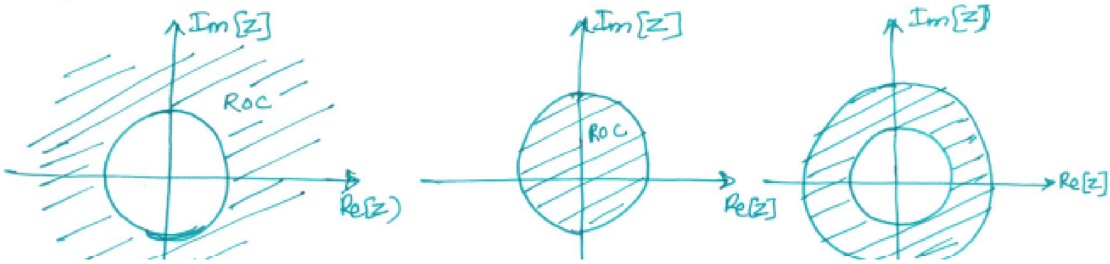
(2) For a finite duration left-sided signal (anticausal signal), the ROC will be entire z-plane except $z=\infty$

(3) For a finite duration two sided signal, the ROC will be the entire z-plane except $z=0$ and $z=\infty$

(4) If $x[n]$ is right sided and of infinite duration (causal signal), then ROC is outside a circle whose radius is equal to the largest pole magnitude (Fig.1)

(5) If $x[n]$ is left sided and of infinite duration (anticausal signal), then ROC is inside a circle whose radius is equal to the smallest pole magnitude (Fig.2)

(6) If $x[n]$ is two sided and of infinite duration, then ROC will be a ring in the z-plane bounded by smallest & largest pole magnitudes (Fig 3)




Source

**(b)** $x(n) = \begin{cases} (-\frac{1}{2})^n; n \leq 1 \\ 0; n \geq 0 \end{cases}$

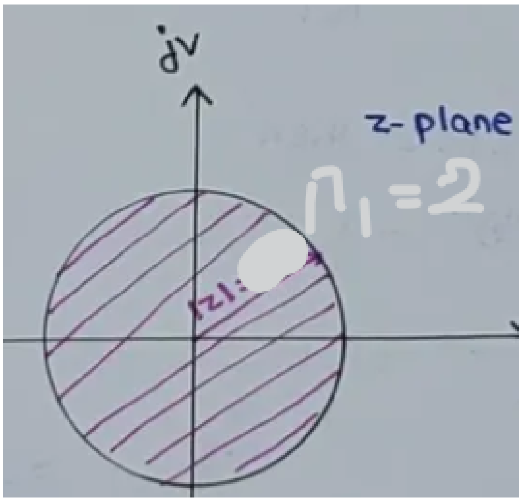
$X(z) = \sum_{n=-\infty}^{1} (-0.5)^n z^{-n} = \sum_{n=-\infty}^{-1} (-0.5z^{-1})^n = \sum_{n=1}^{\infty} (-0.5^{-1}z)^n = \frac{(-0.5^{-1}z)}{1-(-0.5^{-1}z)}$

If $0 < |-0.5^{-1}z| < 1$, then

$X(z) = \frac{(-0.5^{-1}z)}{1-(-0.5^{-1}z)} = -\frac{2z}{1+2z}$

Condition for converges,

$|-0.5^{-1}z| < 1 \Rightarrow |z| < 0.5$



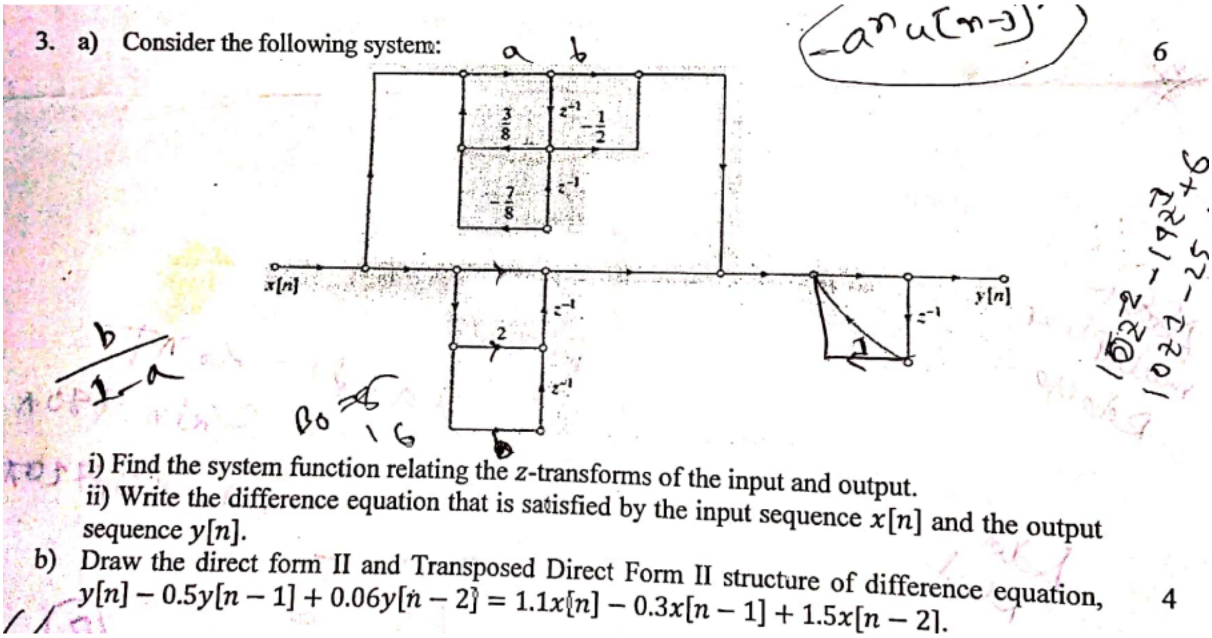

Not 2, it is 0.5

a) Consider an LTI system that is stable and for which $H(z)$, the $z$-transform of the impulse response is $H(z) = \frac{3-7z^{-1}+5z^{-2}}{1-\frac{5}{2}z^{-1}+z^{-2}}$

If $x[n]$ is the input to the system is a unit step sequence. Find the output $y[n]$ by computing the inverse $z$-transform of $Y(z)$

b) Define Cauchy residue theorem. Evaluate the inverse z-transform of $X(z) = \frac{1}{1-bz^{-1}}$ where $|z| > |b|$, using the complex inversion integral.

7. a) Let a system $H(z) = \frac{1-0.5z^{-1}}{1-0.3z^{-1}}$. Describe stability and causality for both the system and inverse system using ROC. show both regions are overlapped. 6



3\. a) Consider the following system: 6



i) Find the system function relating the z-transforms of the input and output.

ii) Write the difference equation that is satisfied by the input sequence $x[n]$ and the output sequence $y[n]$.

b) Draw the direct form II and Transposed Direct Form II structure of difference equation, 4

$y[n] - 0.5y[n-1] + 0.06y[n-2] = 1.1x[n] - 0.3x[n-1] + 1.5x[n-2]$.

- **Differences between Z-transform and Laplace Transform**

| Z Transform | Laplace Transform |
|---|---|
| Z transform is mathematical tool used for conversion of time domain into frequency domain (z domain) and is a function of the complex valued variable Z. | an integral transform that converts a function of a real variable (usually , in the time domain) to a function of a complex variable |
| Used to analyze discrete time signal | continuous time signal |
| Uses the complex variable Z | uses the complex variable s |
| The set of points in z-plane for which X(z) converges is called the ROC of X(z) | The set of points in s-plane for which X(s) converges is called the ROC of X(s) |
| Used for discrete time linear system such as digital filters and sampled data control system | Used for the analysis of CT system such as control system and differential equation |

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$F(s) = \int_0^\infty f(t) e^{-st} t'$$

where $s = \sigma + i\omega$

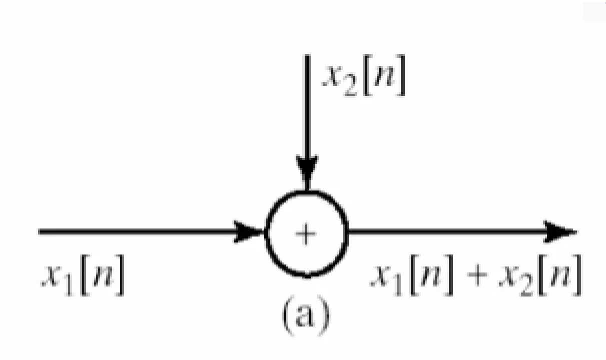
## Digital Filters

A digital filter is a system that performs mathematical operations on a sampled discrete time signal to reduce or enhance certain aspects of that signal.

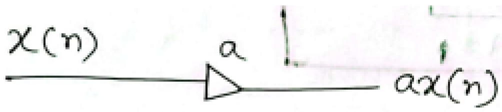
**Types :** FIR, IIR

**Basic element to design Digital Filter**

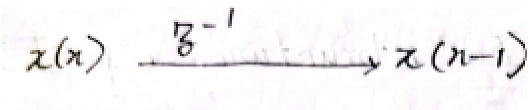
- Adder :




- Multiplexer




- Delay Element




**IIR (Infinite Duration Impulse Response) :** If the impulse response exists infinitely, it is an IIR Filter.

- **Frequency Response of IIR:**

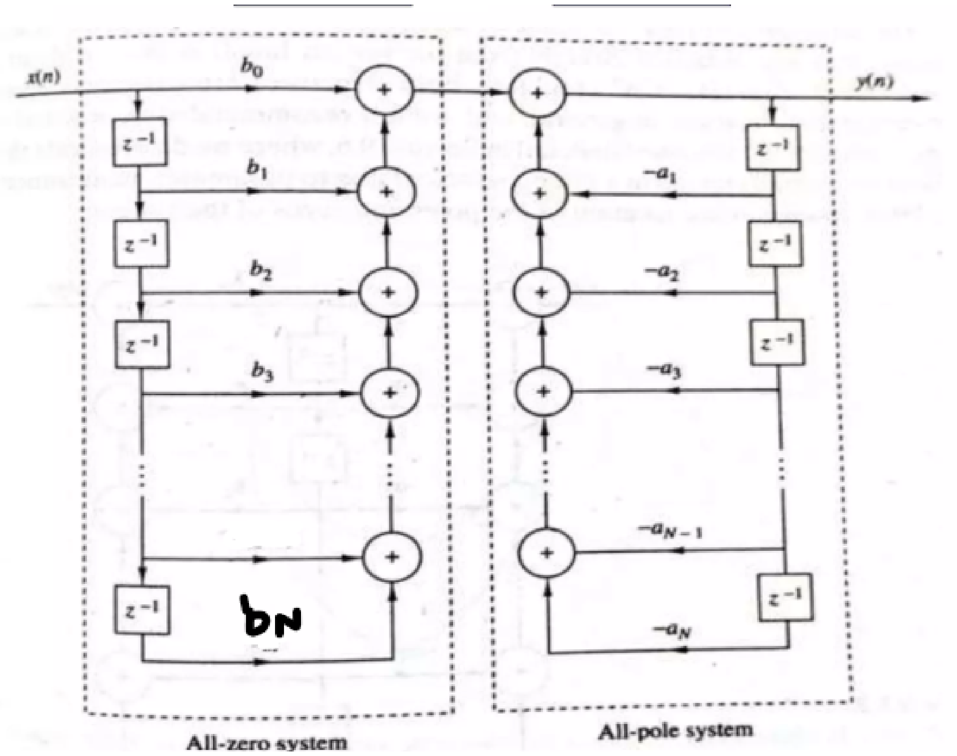
  $H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{1+\sum_{n=1}^{N} a_n z^{-n}}, [a_0 = 1]$

- **Difference equation of IIR Filter**

  $y(n) = \sum_{m=0}^{M} b_m x(n-m) - \sum_{m=1}^{N} a_m y(n-m)$

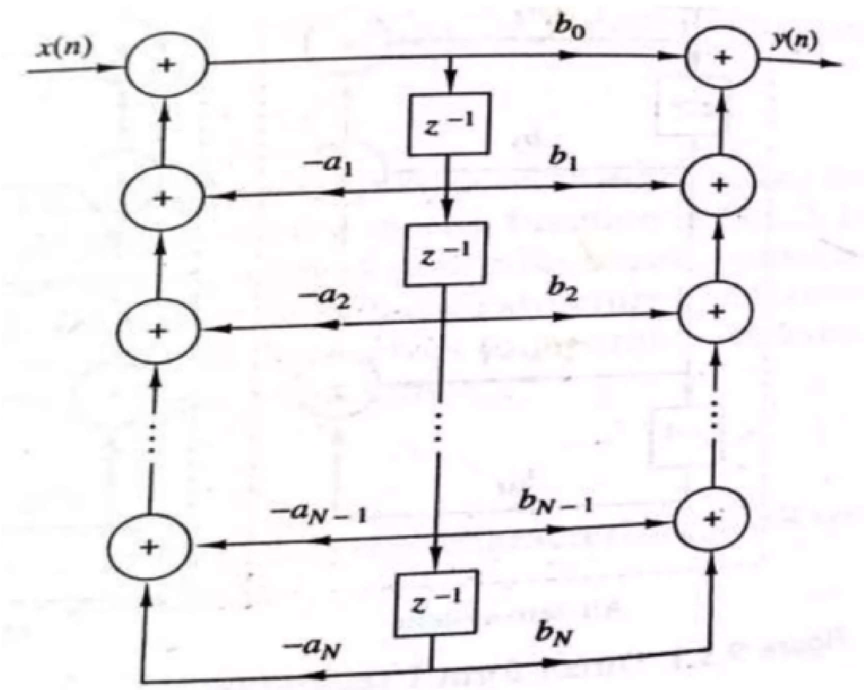
- $N-th$ **order Direct form (I)**

$y(n) = b_0x(n) + b_1x(n-1)... + b_N(n-N) - (a_1x(n-1) + ... + a_Nx(n-N))$




- $N - th$ **order direct form (II)**

  $y(n) = b_0x(n) + b_1x(n-1)... + b_N(n-N) - (a_1x(n-1) + ... + a_Nx(n-N))$



- **Cascade form :**

  In this form the frequency response, $H(z)$ is factored into smaller second section, called bi-quads. The frequency response is then represented as a product of these bi-quads section.

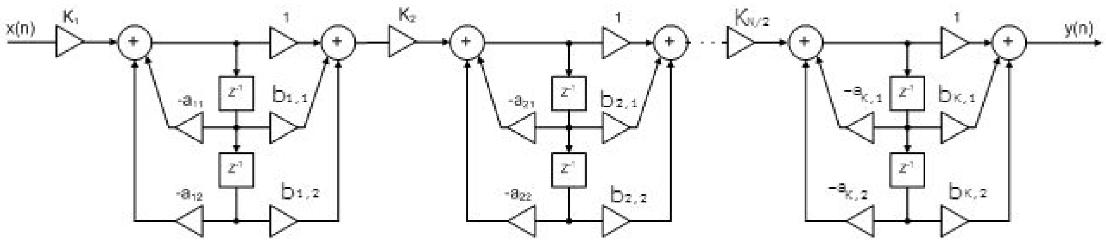
  $H(z) = \frac{B(z)}{A(z)} = b_0 \frac{1+\frac{b_1}{b_0}z^{-1}+....+\frac{b_N}{b_0}z^{-N}}{1+a_1z^{-1}+....+a_Nz^{-N}}$

  $= b_0 \prod_{k=1}^{k} \frac{1+B_{k,1}z^{-1}+B_{k,2}z^{-2}}{1+A_{k,1}z^{-1}+A_{k,2}z^{-2}}$ [for second-order]

  $k = \frac{N}{2}$ [Always]

*Cascade form for $N-th$ order*

$K = \frac{N}{2}$




**FIR (Finite Duration Impulse Response):** The filters have a finite impulse response function which has finite length of time.

- **Transfer Function**

  $M$ : length of filter

  $N$ : Order of filter $= M-1$

  $H(z) = \sum_{n=0}^{M-1} b_n Z^{-n}$

  $h(n) = \begin{cases} b_n, 0 \le n \le M-1 \\ 0, \text{otherwise} \end{cases}$

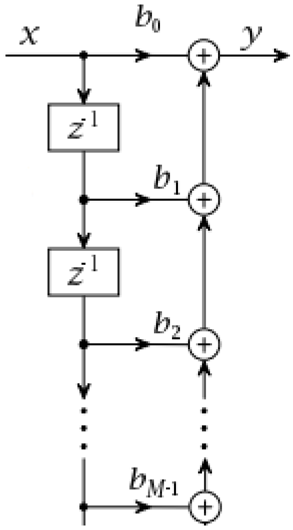
- **Difference equation**

  $y[n] = \sum_{m=0}^{M-1} b_m x(n-m)$

- **Direct form I if $M = M$**

  Order of filter, $N = M-1$

$N-th$ order, $y(n) = b_0x(n) + b_1x(n-1) + ... + b_Nx(n-N)$




- **Cascade FIR**

  For second order section,

  $H(z) = H(z) = \sum_{n=0}^{M-1} b_n Z^{-n} = b_0 + b_1z^{-1} + .... + b_{M-1}z^{-M+1}$

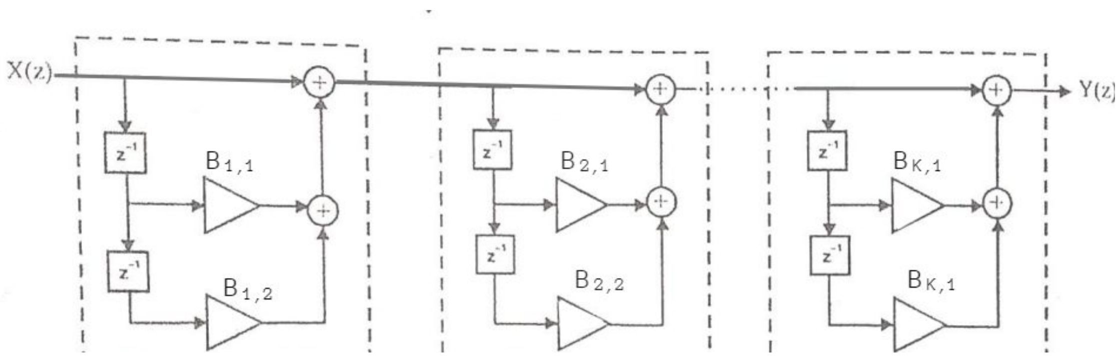
  $= b_0(1 + \frac{b_1}{b_0}z^{-1} + .... + \frac{b_{M-1}}{b_0}z^{-M+1})$

  $= b_0 \prod_{k=1}^{k}(1 + B_{k,1}z^{-1} + B_{k,2}z^{-2})$

  $K = \lfloor \frac{M}{2} \rfloor$

  *Design cascade form of FIR where filter length is* $M$

  $K = \lfloor \frac{M}{2} \rfloor$




- **Properties of Linear-phase FIR filter**

Let, $h(n), 0 < n < M-1$, be the impulse response of length M. The frequency response,

$H(z) = \sum_{n=0}^{M-1} h(n)z^{-n}$

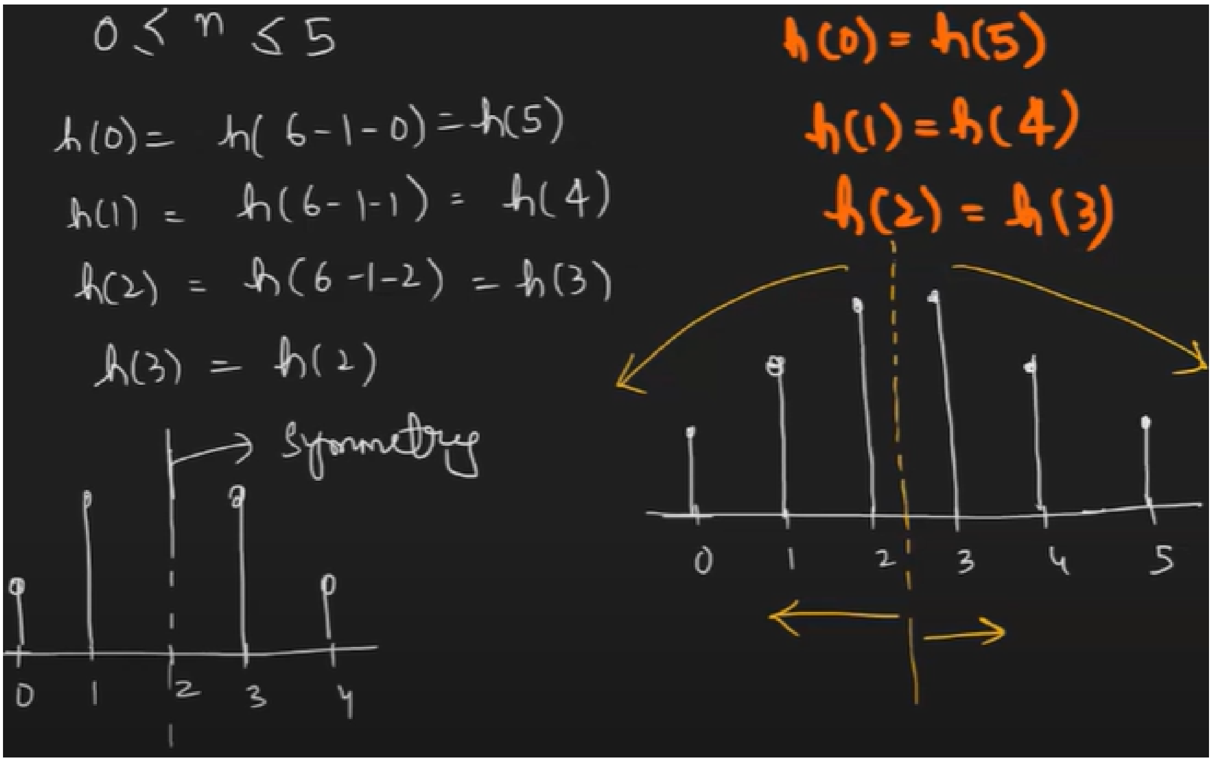
We know, $H(e^{jw}) = \sum_{n=0}^{M-1} h(n)e^{-jwn}, -\pi \leq w \leq \pi$

Where, $\angle H(e^{jw}) = -\alpha w$ (A system has linear phase if its phase response θ(ω) = ∠H(e^jω) = −cω for all ω and any constant c.)

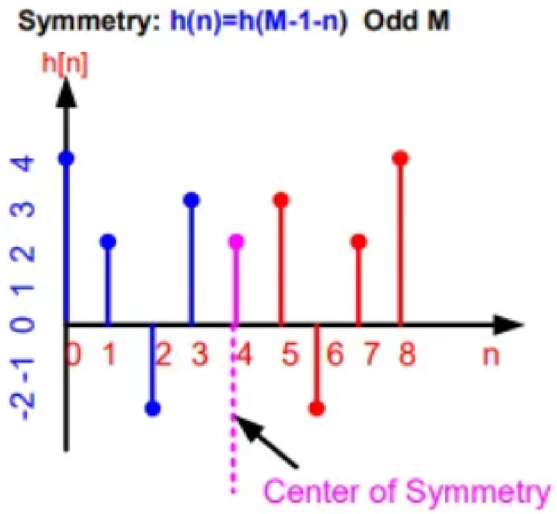
where, $\alpha$ is a constant phase delay, $h(n)$ must be symmetric, if $h(n) = h(M-1-n)$

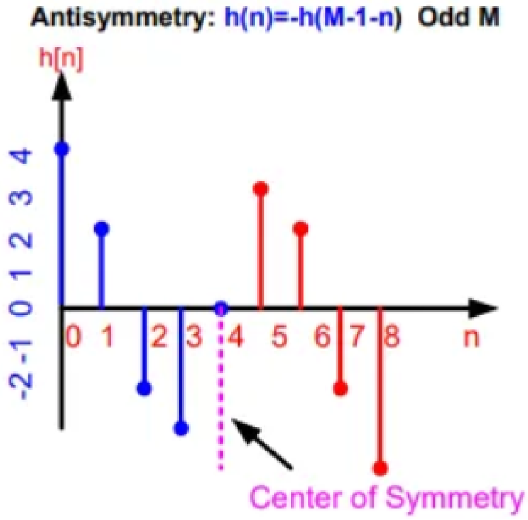
$0 \leq n \leq M-1$ with $\alpha = \frac{M-1}{2}$ if M is odd.

Anti symmetry : $h(n) = -h(M-1-n)$




| b) | Draw the figures of symmetric and anti-symmetric impulse response h(n) of length M. where M is 9. | 2 |
|---|---|---|




**Linear phase system fall into one of 4 categories:**

- **M odd, h[n] is symmetric (Type I)**

  $H(e^{jw}) = \sum_{n=0}^{M} h[n]e^{-jwn}$

  $= h[0](e^{-jw0} + e^{-jwM}) + h[1](e^{-jw1} + e^{-jw(M-1)}) + ... + h[\frac{M}{2}]e^{-jw\frac{M}{2}}$

  $= e^{jwM/2}(h[0](e^{jwM/2} + e^{-jwM/2}) + ... + h[M/2])$

  $= e^{-jwM/2}(h[0].2cos(wM/w) + h[1].2cos(w(M/2-1) + ... + h[M/2])$

  $= e^{-jwM/2}\sum_{k=0}^{M/2} a_1[k]cos(wk)$

  As class note,

  $M = Odd, \alpha = \frac{M-1}{2}$

  $H(w) = \sum_{n=0}^{\frac{M-1}{2}} a(n)cos(wn)$

  $\Rightarrow H(e^{jw}) = e^{-jw\frac{M-1}{2}} \sum_{n=0}^{\frac{M-1}{2}} a(n)cos(wn)$

- **M even, h[n] is symmetric (Type II)**

  $H(e^{jw}) = h[0](e^{-jw0} + e^{-jwM}) + h[1](e^{-jw1} + e^{-jw(M-1)}) + ... + h[\frac{M-1}{2}](e^{-jw\frac{M-1}{2}} + e^{-jw\frac{M+1}{2}})$

$= e^{-jw\frac{M}{2}} \sum_{n=0}^{\frac{M-1}{2}} b(n)cos(w(k+\frac{1}{2}))$

As class note,

$H(e^{jw}) = e^{-jw\frac{M-1}{2}} \sum_{n=1}^{\frac{M}{2}} b(n)con\{w(n-\frac{1}{2})\}$

- M even, h[n] is antisymmetric (Type III),
- M odd, h[n] is antisymmetric (type IV)