Electromagnetic Wave and Radiating System CT - 1

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Electromagnetic Wave: Electromagnetic Waves or EM waves are waves that are created as a **result of vibrations an electric field and a magnetic field**.

Microwave: Microwave are defined as electromagnetic radiations with a **frequency ranging between 300MHz to 300 GHz.** Example : Satellite communication signal.

Del Operator / Nabla Operator

Also known as vector differential operator.

- collection of partial derivative operators
- a metatheatrical operator commonly used in vector calculus
- “Operator” is similar to a function
- It’s defined as a vector, but it doesn’t have magnitude. So, it isn’t a “true” vector.

$$\nabla = \frac{\delta}{\delta x}i + \frac{\delta}{\delta y}j + \frac{\delta}{\delta z}k$$

Uses of Del operator

- **Gradient :** The gradient of a scalar field is a vector that points in the direction in which the field is most rapidly increasing, with the scalar part equal to the rate of change.

  $$Gradient = \vec{\nabla}\vec{V}$$

- **Divergence:** A divergence shows how the field behaves towards away from a point. The divergence of a vector field is scalar field.

  $$Divergence = \vec{\nabla}.\vec{V}$$

- **Curl:** A curl is used to measure the rotational extent of the field about a particular point. The curl of a vector field is a vector field.

  $Curl = \vec{\nabla} \times \vec{V}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

The Divergence Theorem :

The divergence theorem states that the surface integral of the normal component of a vector point function F over a closed surface S is equal to the volume integral of the divergence of $\vec{F}$

$$\int_x \int_y \int_z \vec{\nabla}.\vec{F}\, d\vec{v} = \int_v \vec{\nabla}.\vec{F}\, d\vec{v} = \oint_s \vec{F}\ .\ d\vec{s}$$

Example: Compute ∬S F dS, where F = $(3x + z, y^2 - \sin x^2\, z, xz + ye^{x^5})$,
where 0≤x ≤1, 0 ≤ y ≤3, 0 ≤ z ≤ 2

Solution:

div F = 3 + 2y + x.

∬s F dS =∭B div F dV

Given interval is: 0≤x ≤1, 0 ≤ y ≤3, 0 ≤ z ≤ 2

Now substitute the intervals in the lower and the upper limits of the integrals, then we have:

∬s F dS = $\int_0^1 \int_0^3 \int_0^2 (3 + 2y + x)\,dz\,dy\,dx$

Integrate the function with respect to z and substitute the limits.

= $\int_0^1 \int_0^3 (6 + 4y + 2x)\,dy\,dx$

Now, integrate with respect to x and y, and substitute the limits, we get:

= 36 + 3

= 39

Thus, ∬S F dS = 39.

Stokes Theorem

Statement: The surface integral of the curl of a function over a surface bounded by a closed surface is equal to the line integral of the particular vector function around the surface.

$$\oint_C \vec{F}\vec{dr} = \int\int_s \vec{F} \times \vec{\nabla}\ \vec{ds}$$

Using stokes theorem, evaluate:

$\int\int_S curl\vec{F}.d\vec{S},\ where\vec{F} = xz\hat{i} + yz\hat{j} + xy\hat{k},$
such that S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

Solution:

Given,

Equation of sphere: $x^2 + y^2 + z^2 = 4$....(i)

Equation of cylinder: $x^2 + y^2 = 1$....(ii)

Subtracting (ii) from (i),

$z^2 = 3$

$z = \sqrt{3}$ (since z is positive)

Now,

The circle C is will be: $x^2 + y^2 = 1$, $z = \sqrt{3}$

The vector form of C is given by:

$r(t) = cost\hat{i} + sint\hat{j} + \sqrt{3}\hat{k}; 0 \le t \le 2\pi$

$Thus,\ r'(t) = -sint\hat{i} + cost\hat{j}$

Let us write F(r(t)) as:

$F(r(t)) = \sqrt{3}cost\ \hat{i} + \sqrt{3}sint\ \hat{j} + cost\ sint\ \hat{k}$

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_C \mathbf{F}\cdot d\mathbf{r}\\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt\\ &= \int_0^{2\pi}\left(-\sqrt{3}\cos t\sin t + \sqrt{3}\sin t\cos t\right)dt\\ &= \sqrt{3}\int_0^{2\pi} 0\,dt\\ &= 0\end{aligned}$$

Electric Field

- a vector field
- the force per unit charge exerted/applied on a positive test charge at rest at a specific point

$$E = \frac{F}{q}$$

Electric Displacement Vector

- is the charge per unit area that would be displaced across a layer of conductor placed across an electric field
- denoted by D

$D = \epsilon_0 E + P$

$\text{Polarization density, P } = \frac{\Delta p}{\Delta V}$

$\Delta p = \text{dipole moment}$
$\Delta V = \text{volume element}$

Isotropic medium

- all properties are **uniform and independent of direction**
- does not depend on the direction
- the velocity of light is the same in all direction
- Example : Glass

Anisotropic medium

- the properties are different in all direction
- direction-dependent
- the velocity of light differs according to direction

Ampere's Circuit Law: The line integral of Magnetic field intensity along a closed path is equal to the current distribution passing through that loop.

$\oint H \cdot dL = I_{enc} = \iint J \cdot ds$

$J = \text{current density}$

Maxwell Four Equation

<u>**Maxwell first equation:**</u> $\vec{\nabla}.\vec{D} = \rho_v$

Statement : The total electric displacement through the surface enclosing a volume is equal to the total charge with the volume. The more charge density the more

Electric field.

$\oint \vec{D} \cdot d\vec{s} = Q_{enc}$ ...(1)

$\oiint \vec{D}.d\vec{s} = \iiint \nabla.\vec{D}d\vec{v}$ ...(2)[from Divergence theorem]

$\iiint \nabla.\vec{D}d\vec{v} \ = Q_{enc}$ ..(3)[from equation 1 and 2]

The volume charge density,

$\rho_v = \frac{dQ}{dv}$
or, $dQ = \rho_v dv$
or, $Q = \iiint \rho_v dv$

$\iiint \vec{\nabla}.\vec{D}d\vec{v} = \iiint \rho_v d$
$\nabla.D = \rho_v$

Maxwell Second equation $\vec{\nabla} \cdot \vec{B} = 0$

Statement: The total outward flux of magnetic induction (B) through any closed surface is equal to zero.

From Gauss's Law, $\oiint \vec{B} \cdot d\vec{A} = 0$

$\oiint \vec{B} \cdot d\vec{A} = \int_v \vec{\nabla}.\vec{B}d\vec{v} = \vec{\nabla}.\vec{B} = 0$

Maxwell Third equation: $\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$

Statement: The electromotive force around a closed path is equal to negative rate of change of magnetic flux linked with the path. That means change of magnetic flux will create the electric field.

Faraday's Law, $\epsilon_{emf} = -\frac{d\phi}{dt}$

Gauss Law, $\epsilon_{emf} = \oint_c \vec{E}.dl$

$\oint_c \vec{E}.dl = -\frac{d\phi}{dt} = \frac{d\oint_S -\vec{B}.da}{dt} = -\oint_S \frac{d\vec{B}}{dt}da$

or, $\oint_S \vec{\nabla} \times \vec{E} \ da = -\oint_S \frac{d\vec{B}}{dt}da$

or, $\oint_S (\vec{\nabla} \times \vec{E} + -\frac{d\vec{B}}{dt}) \ da = 0$

or, $\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$

Maxwell Fourth Equation or Maxwell-Ampere's Law: $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t}$

Statement: The magnetomotive force around a closed path is equal to the summation of conductor current and displacement current through any surface bounded by the path. Changing electric field creates magnetic field.

Ampere's circuit law, $\oint_c \vec{B}.\vec{dl} = \mu_0 i$

By Stroke's theorem, $\oint_c \vec{B}.\vec{dl} = \oint_S (\vec{\nabla} \times \vec{B})\vec{dS}$

$\oint_S (\vec{\nabla} \times \vec{B})\vec{dS} = \mu_0 i = \mu_0 \oint_S \vec{J}.\vec{dS}$

or, $\oint (\vec{\nabla} \times \vec{B} - \mu_0 \vec{J})\vec{dS} = 0$

or, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

or, $\vec{\nabla} \times \vec{H} = \vec{J}$

Modified Ampere's circuit law,

$\oint_c \vec{B}.\vec{dl} = \mu_0 i + i_d\ [i_d : \text{displacement current}]$

$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d = \vec{J} + \epsilon_0 \frac{\delta \vec{E}}{\delta t} = \vec{J} + \frac{\delta \vec{D}}{\delta t} \ \ [E = \frac{D}{\epsilon_0}]$

Derive equation of continuity by Maxwell's 4th equation

$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\delta D}{\delta t}$

or, $0 = \vec{\nabla}.(\vec{J} + \frac{\delta D}{\delta t})$

or, $\vec{\nabla}.\vec{J} = -\frac{\delta \rho}{\delta t}$

Why Ampere's Law is not correct ?

Ampere's Law is valid only for steady current or when the electric field does not change with time.

$\vec{\nabla}.\vec{J} = -\frac{\delta \rho}{\delta t}$, equation of continuity

From Maxwell fourth equation, $\vec{\nabla} \times \vec{H} = \vec{J}$

or, $\vec{\nabla}.(\vec{\nabla} \times H) = \vec{\nabla}.\vec{J}$

or, $\vec{\nabla}.\vec{J} = 0,$ [$\vec{\nabla}(\vec{\nabla} \times H) = 0$] it doesn't match with continuity equation.

Drawback of Ampere's Law

- applicable for constant current or when the **electric field does not vary the time**
- valid only for the points where there is **only conduction current** and no displacement current
- doesn't match with continuity equation

<u>Problems:</u>

- **Type 1:** Electron and proton number is given and you have to calculate the electric flux.

  Find the electric flux through the surface of a sphere containing m protons and n electrons.

  $\text{Flux}, \phi_E = \frac{q_{enc}}{\epsilon_0},$

  $q_{enc} = m \cdot q_p + n \cdot q_e$

  $q_p = 1.6 \times 10^{-19}, q_e = -1.6 \times 10^{-19}, \epsilon_0 = 8.85 \times 10^{-12}$

- **Type 2:** Surface Charge density and x, y is given

  A cube of side L contains a flat plate with variable surface charge density, $\rho = -3xy$. If the plate extends from $x = \ 0 \text{ to L}$ and $y = \ 0 \text{ to M}$. What is the total electric flux through the wall of the cube ?

  $\phi_E = \frac{q_{enc}}{\epsilon_0}$

  $q_{enc} = \int_s \rho da = \int_{y=0}^{M} \int_{x=0}^{L} -3xy \, dxdy$ = … [Applicable for continuous charge]

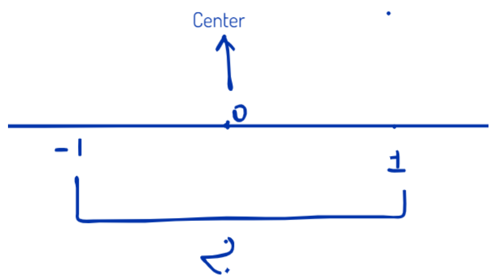
- **Type 3:** Electric Flux density is given and find the charge density ρ

  The electric flux density is given as, $\vec{D} = x^3 \hat{i} + x^2 y \hat{j}$. Find the charge density ρ inside a cube of side $2m$ placed centered at the origin with its side along the co-ordinate axes.

$\vec{\nabla}.\vec{D} = \rho$

or, $\rho = (\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k})(x^3\hat{i} + x^2y\hat{k}) = 3x^2$

Charge, $Q = \int_v \rho dv = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 3x^2 dxdydz = 8C$



- **Type 4:** Sphere

$dv = r^2 sin\theta dr d\theta d\phi$ $[0 < r < 2,\ 0 < \theta < \pi, 0 < \phi < 2\pi]$

$\int uvdx = u\int vdx - \int(u'\int vdx)dx$

The volume charge density inside a hallow sphere is $\rho = 10e^{-30\pi}\text{cm}^{-2}$. Find the total charge enclosed within the sphere. Also find electric flux density on the surface of the sphere.

$Q = \int_v \rho dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{2} 10e^{-20\pi} r^2 sin\theta dr d\theta d\phi = \frac{\pi}{100}$

$D = \epsilon.E = \frac{1}{4\pi r^2}\cdot\frac{\pi}{100} = \frac{1}{400r^2}$

- **Type 5:**

In a conducting medium the magnetic field is given as $H = y^2x\hat{i} + 2(x + 1)yz\hat{j} - (x+1)z^2\hat{k}$. Find the conduction current density (J) at point $(2, 0, -1)$; Also find the current $y = 1, 0 < x < 1, 0 < z < 1$.

$$J = \vec{\nabla} \times \vec{H} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y^2x & 2(x+1)y^3 & -(x+1)z \end{bmatrix}$$

$= -(2xy + 2y)\hat{i} + (z^2 + y^2)\hat{j}$

At point $(2, 0, -1)$, $J = \hat{j}$

At $y = 1$, $\vec{J} = (1 - z^2)\hat{j}$

$$I = \int_S \vec{J} dA = \int_z \int_x J dx dy = \int_0^1 \int_0^1 \hat{(}1 + z^2) dx dz = \frac{2}{3} A$$

- **Type 6:** EMF

  Find the emf induced in a square loop with sides of length lying in the yz plane is a region in which magnetic field charge overtime, $B(t) = B_0 e^{-5t/t_0} \hat{i}, \epsilon = \frac{-d}{dt} \int_s \vec{B} \, \hat{\eta} \, \vec{d}a, \text{unit normal } \hat{\eta} = \hat{i}.$

$$\begin{aligned}
&\text{emf} = -\frac{d}{dt} \int_s \vec{B} \cdot \vec{\eta} \, d\vec{a} = -\frac{d}{dt} \int_s B_0 e^{-5t/t_0} \hat{i}.\hat{i} d\vec{a} \\
&= -\frac{d}{dt} B_0 e^{-5t/t_0} \int_s da \\
&= -\frac{d}{dt} B_0 e^{-5t/t_0} a^2 \\
&= \frac{5a^2 B_0}{t_0} e^{-5t/t_0}
\end{aligned}$$