Theory of Computation

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NB: There are many mistakes. **Read at your own risk.**

What is TOC?

- A branch of theoretical CS
- Whether and how efficiently a problem can be solved on computational model, using an algorithm

TOC has major 3 branches.

- Automata theory: deals with various definition and properties of mathematical model
- Computability theory: what can and cannot be computed
- Computational Complexity theory: it groups computable problem based on hardness

Model of Computation: mathematical abstraction of Computer

Definition: describe a object and notations.

Theorem: mathematical statement basis on previously stablished statement.

Proof: convincing logical argument that statement is true.

Lemma: A minor result (theorem, it's the lemma) to prove another theorem.

Corollaries: A result in which the proof relies heavily.

THEOREM

If m and n are any two whole numbers and

- $a = m^2 - n^2$
- $b = 2mn$
- $c = m^2 + n^2$

then $a^2 + b^2 = c^2$

Proof:

$$\begin{aligned} \mathbf{a^2 + b^2} &= (m^2 - n^2)^2 + (2mn)^2 \\ &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \\ &= (m^2 + n^2)^2 \\ &\mathbf{= c^2} \end{aligned}$$

Deductive Proof: If H (hypothesis), then C (conclusion)

- Sequence of statement
- hypothesis to conclusion

Contradiction Proof:

- Assume the theorem is false, this assumption leads to false
  Example: sqrt (2) is irrational.

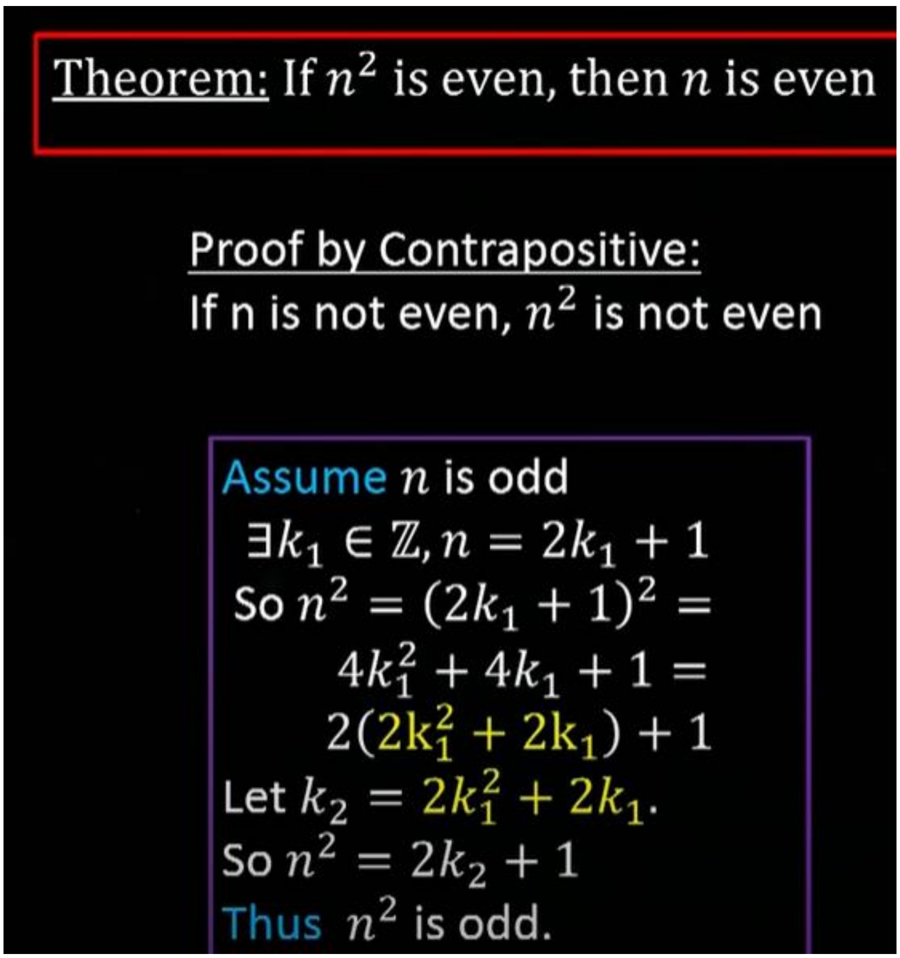
Induction proof:

- All elements of infinite set have a specific property
- 2 Steps
  - Base case

- Induction Step (Assume S(k), then S(k+1))
- Example: Σn = n(n+1)/2

Contraposition Proof:

- P=>Q is equivalent to ¬Q => ¬P
- Assume ¬Q is true, prove ¬P is true
- Example : If n is even, n^2 is even



Counter Example proof:

- Show an example to disprove the claim

Automata theory

- Study of abstract machines and computational problems that can be solved by these machines

Automata:

- abstract machine/computing device
- mathematical model of a system that involves with input, output, state, transition etc.

Consists of

- States: Circle, description of the status of the system
- Transition: Arrow, input, one state to another

Basic Definition:

1. Symbols
   a. Symbols are indivisible objects or entity that cannot be defined.

2. Alphabets
   a. Finite set of symbols
   b. Σ

3. String
   a. Finite sequence of symbols
   b. Denoted by w,z,y,z

4. Empty String
   a. Denoted by ε
   b. The length of the empty string is 0

5. Length of a string
   a. Denoted by |W|

6. Power of Alphabets
    a. Set of string length k, Σ^k
        i. $\Sigma^0 = \{\varepsilon\}$
    b. Set of all string including empty, $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \dots \Sigma^n$
    c. Set of all String w/o empty string, $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \dots \cup \Sigma^n$

7. Concatenation of a string
    a. x = 01, y=10 concatenation of x and y, xy = 0110, yx = 1001
8. Language
    a. A language over an alphabet is a set of strings over that alphabet.
    b. Set of all Σ^*
    c. Empty language Φ

Membership Problem:

- Given a word and a Language, we want to check word belongs to the language or not, this is called membership problem.

3 requirements of automata:

- Taking input
- Producing output
- May have Temporary storage
- Control unit: can change state according to transition function

Finite automata

- An abstract computing device
- No temporary storage
- Used to recognize pattern
- Accept or reject input depending on pattern

2 types

- DFA (Deterministic Finite Automata)
- NFA (Non-Deterministic Finite Automata)

DFA	NFA
one state transition in DFA	May have more than one
Cannot ε	Can use ε
Understand as one machine	Multiple machine
have max. one possible next state **for one input**	May have multiple next possible states for one input
Difficult to construct	Easier
Time less	Executing time more
All DFA = NFA	All NFA != DFA
$\delta: Q \times \Sigma \rightarrow Q$	$\delta: Q \times \Sigma \rightarrow 2^Q$

State:

- Description of the Status of system waiting to execute a transition
- Denoted by Circle/Vertex

Transition:

- set of actions to execute when a condition is fulfilled or an event received.
- Denoted by Arrow/Edge

DFA

`Deterministic finite automata (or DFA) are finite state machines that accept or reject strings of characters by parsing them through a sequence that is` **`uniquely determined by each string.`**

- A formalism for defining languages, consisting of:
  1. A finite set of *states* (Q, typically).
  2. A finite set of *input symbols* (Σ, typically).
  3. A *transition function* (δ, typically).
  4. A *start state* (q_0, one of the states in Q, typically).
     1. Only one start state
  5. A set of *final states* ($F \subseteq Q$, typically).
     - "Final" and "accepting" are synonyms.
     - May have multiple final states
- So, A DFA is a *five-tuple* notation:

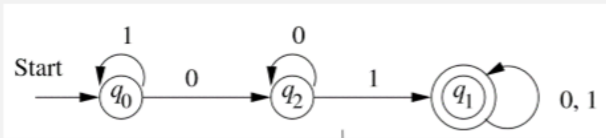
$$A=(Q, \Sigma, \delta, q_0, F)$$

where ***A*** is the name of the DFA.

From Nazia ma'am's Slide

Lecture 3:

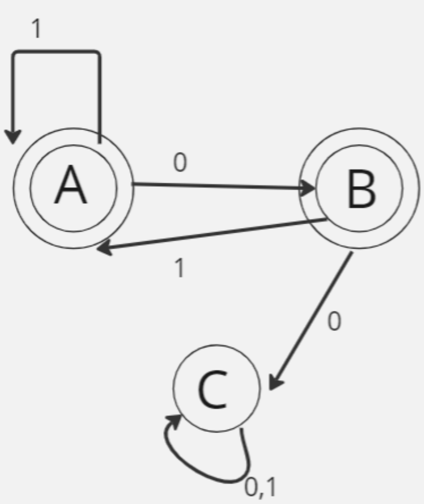
1. Show a transition table for the following diagram/automata:



→ q0	q2	q1
q2	q2	q1
*q1	q1	q1

2. Draw a transition diagram from the following table:

	0	1
→ [illegible]	[illegible]	[illegible]
$*B$	C	A
C	C	C



Extended Transition Functions

- Denoted by $\hat{\delta}$
- Takes state q and string w (where Transition function usually takes only alphabet)
- Induction steps

A recursive algorithm is used to reach the final state, which is as follows:

1. Base condition:

$$\hat{\delta}(q, \epsilon) \rightarrow q$$

2. Recursion rule:

$$\hat{\delta}(q, xa) \rightarrow \delta(\hat{\delta}(q, x), a)$$

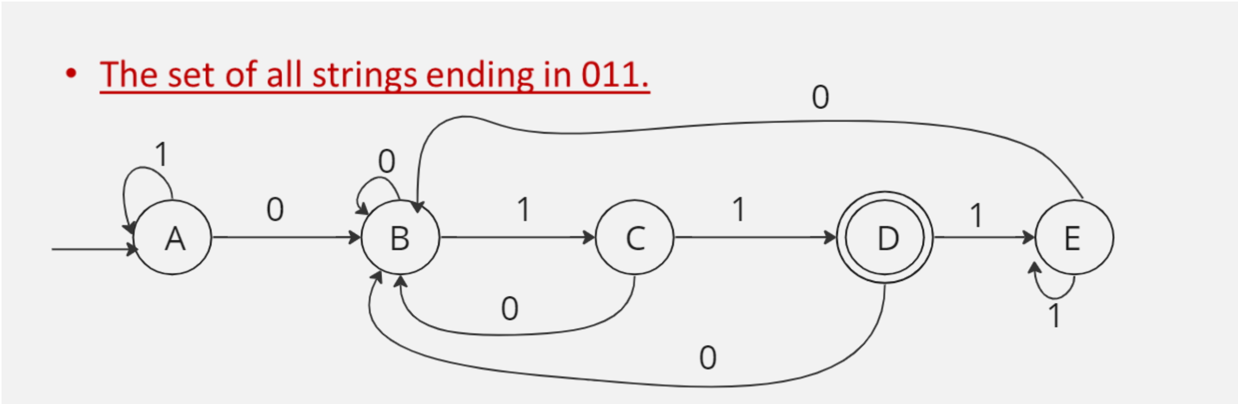
Here, $x \in \Sigma^*$ and $a \in \Sigma$. Also,x is a string of characters belonging to the set of the input symbols and a is a single character.

$$* \quad \hat{\delta}(q_0, 110101)$$
$$= \delta(\hat{\delta}(q_0, 11010), 1)$$
$$= \delta(\delta(\hat{\delta}(q_0, 1101), 0), 1)$$
$$= \delta(\delta(\delta(\hat{\delta}(q_0, 110), 1), 0), 1)$$
$$= \delta(\delta(\delta(\delta(\hat{\delta}(q_0, 11), 0), 1), 0), 1)$$
$$= \delta(\delta(\delta(\delta(\delta(\hat{\delta}(q_0, 1), 1), 0), 1), 0), 1)$$
$$= \delta(\delta(\delta(\delta(\delta(q_1, 1), 0), 1), 0), 1)$$
$$= \delta(\delta(\delta(\delta(q_0, 0), 1), 0), 1)$$
$$= \delta(\delta(\delta(q_2, 1), 0), 1)$$
$$= \delta(\delta(q_3, 0), 1)$$

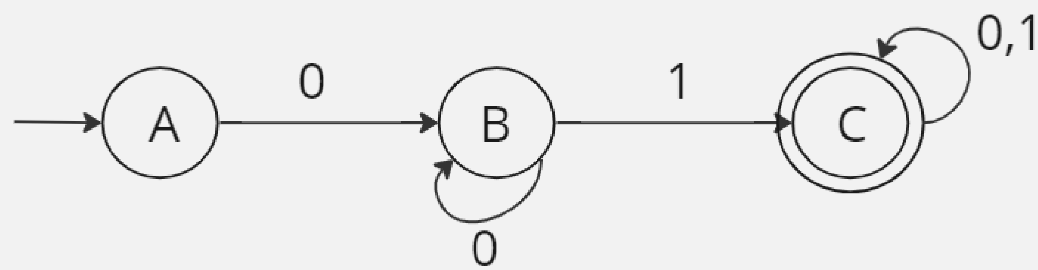
$$= \delta(q_1, 1)$$
$$= q_0$$

From : Pollock bhai

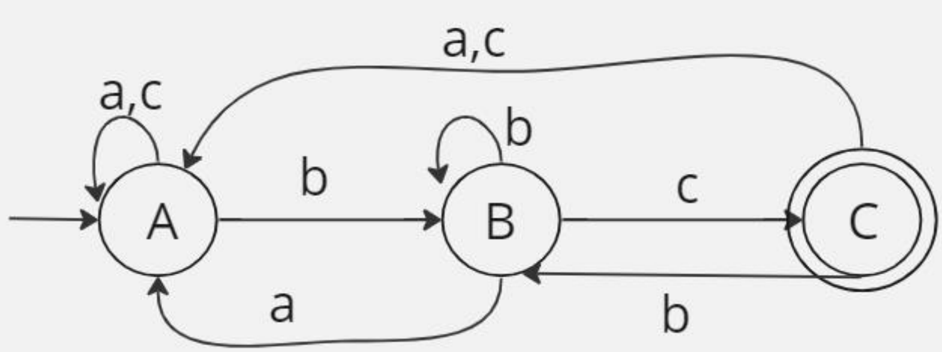
Lecture 4:



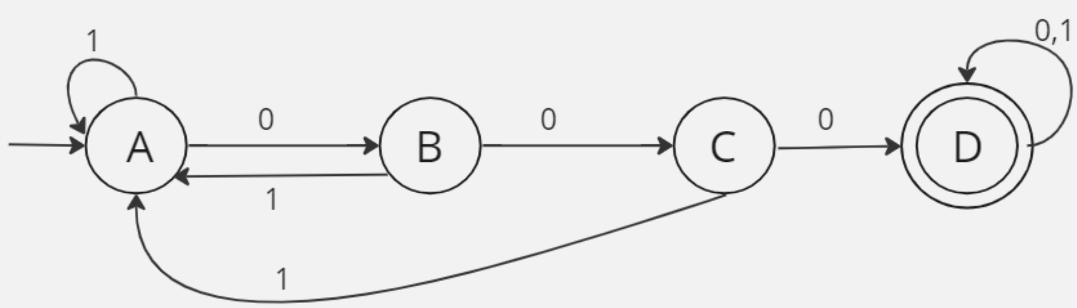
- The set of all string with 01 as a substring.



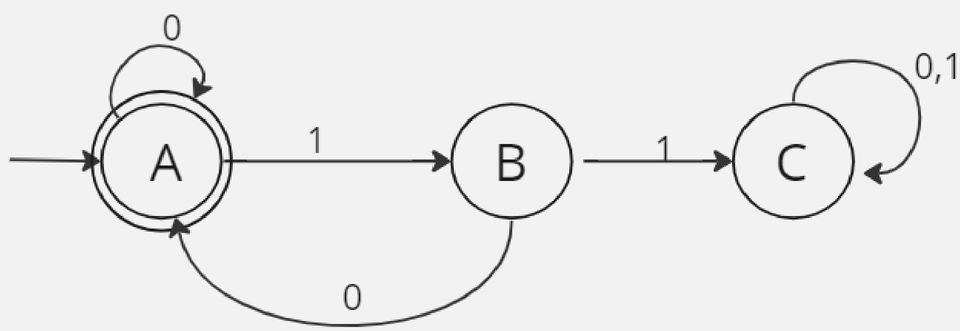
- Draw a DFA for the following language over $\Sigma = \{a,b,c\}$
  - The set of all strings ending in bc.



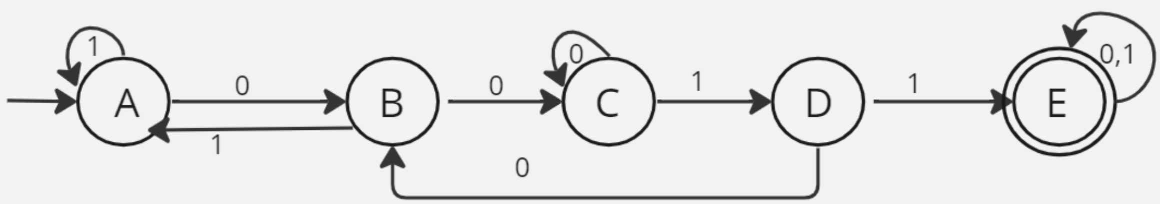
- The set of all strings with three consecutive 0's (not necessarily at the end).



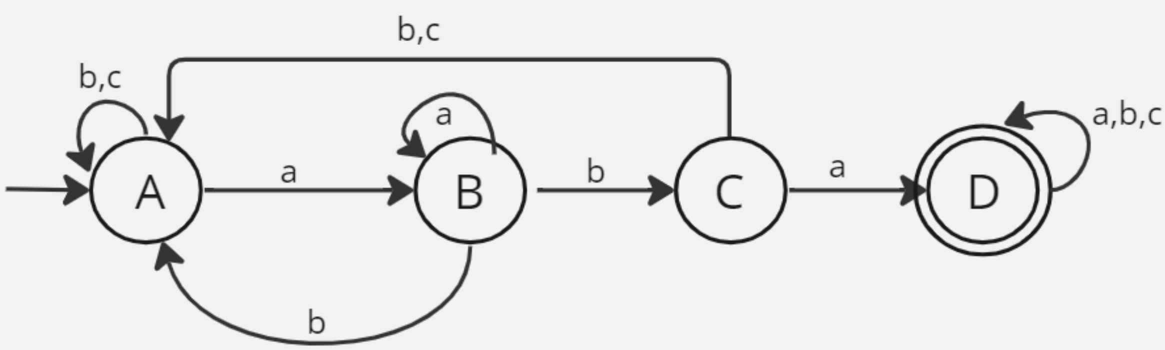
- The set of all strings with not having two consecutive 1's (not necessarily at end).



- The set of strings containing two consecutive zero's followed by two consecutive ones.



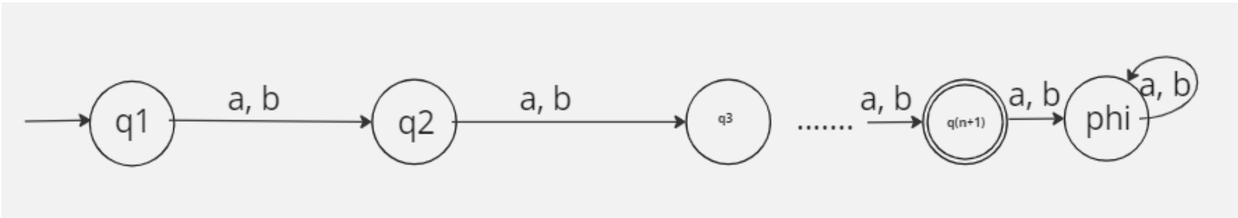
- The set of all strings with aba is a substring.



Lecture: 5

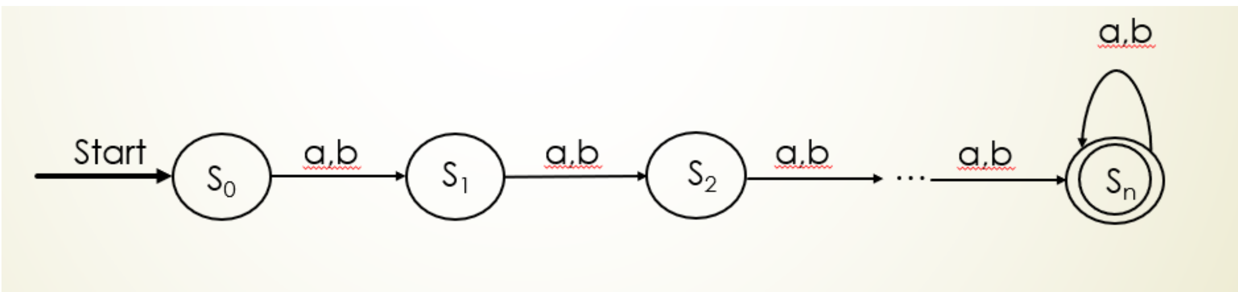
DFA with exactly n alphabets

i. Number of states: n+2



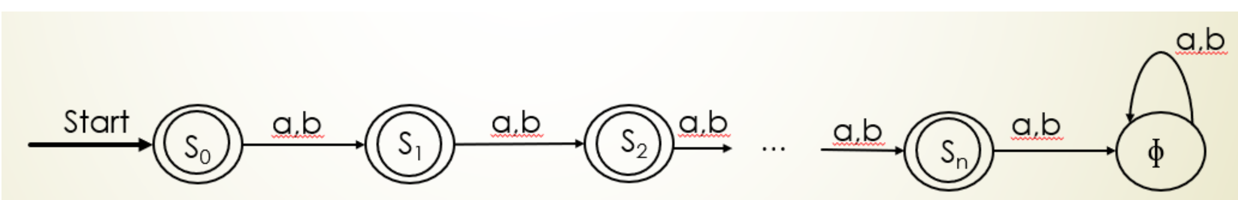
DFA with at least n alphabets

i. Number of states: n+1

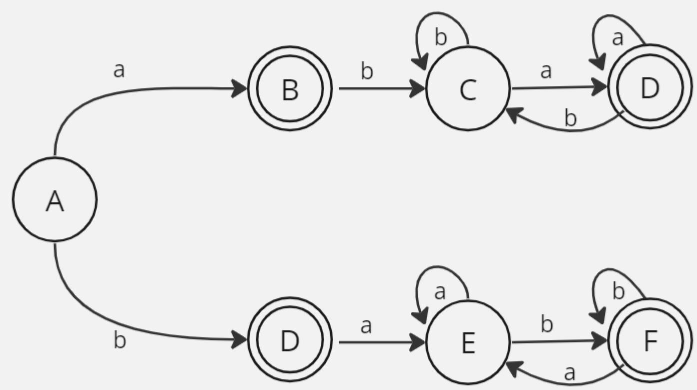


DFA with at most n alphabets

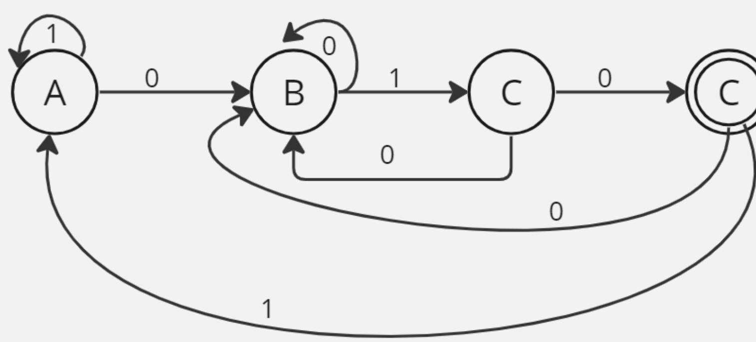
i. Number of states: n+2



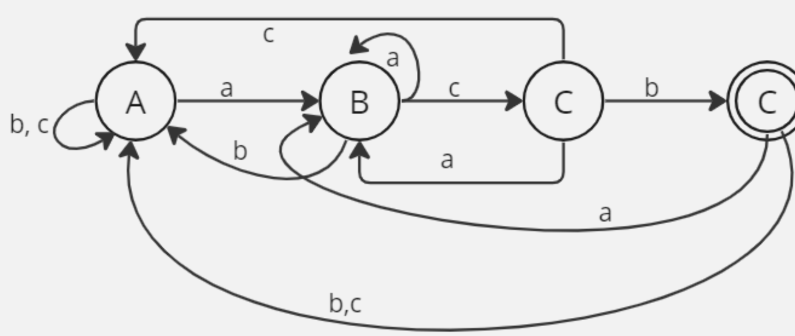
➨ Draw a DFA that accepts equal number of **ab** and **ba** over Σ= {a,b}.



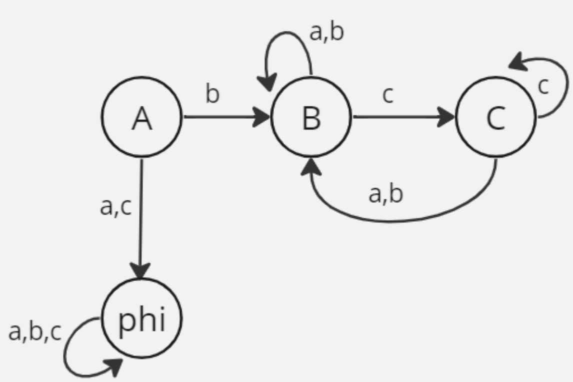
➨ Draw a DFA for the set of all strings ending in **010** over Σ= {0,1}.



➨ Draw a DFA for the set of all strings ending in **acb** over Σ= {a,b,c}.



➨ Draw a DFA for the set of all strings starting with **b** and ending with **c** over Σ= {a,b,c}.



Minimizing DFA

- Equivalence Theorem

- Using equivalence theorem, we have
  - 0 equivalence

    P_0={A,D,E} {B,C,G}
  - 1 equivalence

    P_1= {A,D,E} {B,C}{G}
  - 2 equivalence

    P_2={A} {D,E} {B,C} {G}
  - 3 equivalence

    P_3={A} {D,E} {B,C} {G}
  - Since P_3=P_2, we stop.

	0	1
→A	B	C
*B	D	E
*C	E	D
D	G	G
E	G	G
*G	G	G

	0	1
→{A}	{B,C}	{B,C}
{D,E}	{G}	{G}
*{B,C}	{D,E}	{D,E}
*{G}	{G}	{G}

Table Filling Method / Myhill-Nerode Theorem

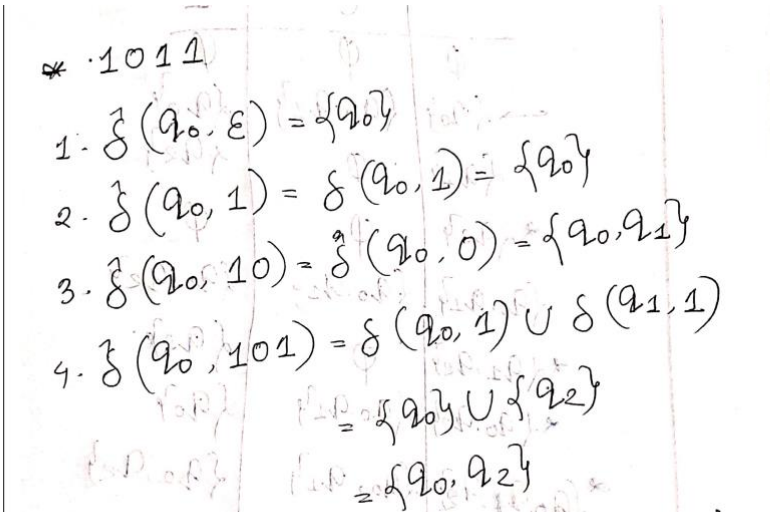
Steps:

1) Draw a table for all pairs of states (P,Q)
2) Mark all pairs where P∈ F and Q∉F
3) If there are any Unmarked pairs (P,Q) such that [δ(P,x), δ(Q,x)] is marked, then mark [P,Q] where 'x' is an input symbol
   REPEAT THIS UNTIL NO MORE MARKINGS CAN BE MADE
4) Combine all the Unmarked Pairs and make them a single state in the minimized DFA

NFA

- Non-deterministic Finite Automata
- Transition from a state on an input symbol can be to any set of states
- DFA: Q x Σ→ Q
- NFA: Q x Σ→ 2^Q
- No need to add dead state/trap state
- Input can be empty

Extended Transition for NFA



* 1011

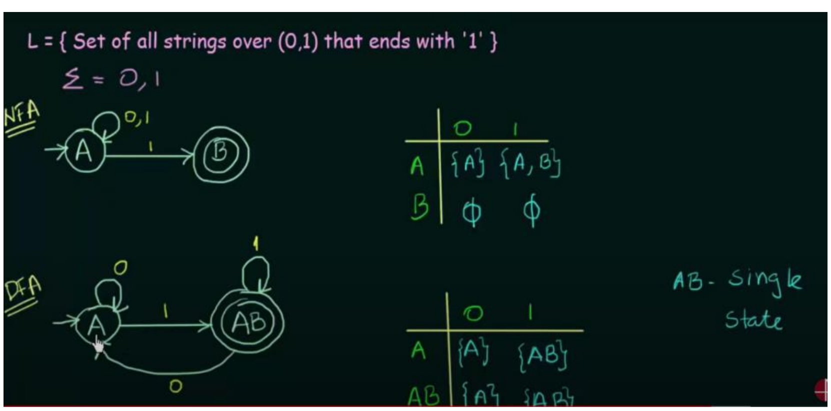
1. $\hat{\delta}(q_0, \varepsilon) = \{q_0\}$
2. $\hat{\delta}(q_0, 1) = \delta(q_0, 1) = \{q_0\}$
3. $\hat{\delta}(q_0, 10) = \hat{\delta}(q_0, 0) = \{q_0, q_1\}$
4. $\hat{\delta}(q_0, 101) = \delta(q_0, 1) \cup \delta(q_1, 1)$

$= \{q_0\} \cup \{q_2\}$

$= \{q_0, q_2\}$

From: Pollock bhai

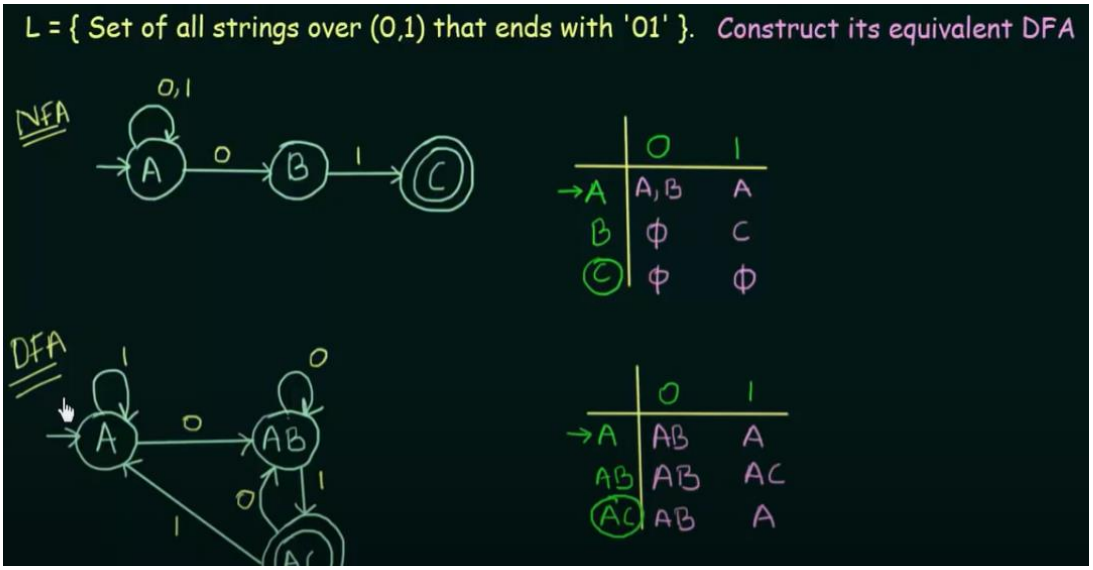
Conversion of NFA to DFA:



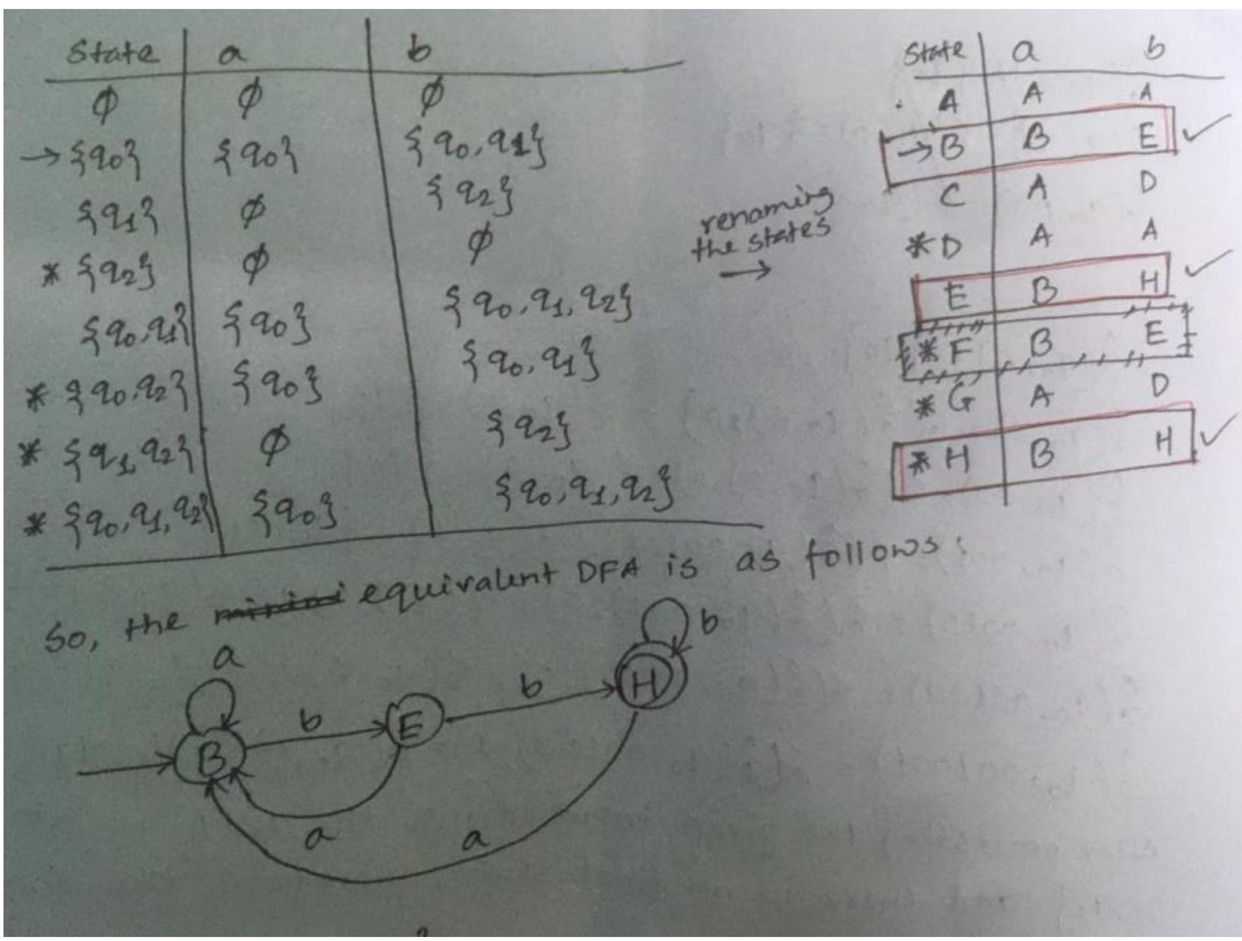
	0	1
A	{A}	{A, B}
B	ϕ	ϕ

	0	1
A	{A}	{AB}
AB	{A}	{AB}

If there are any empty state/phi in NFA, a dead state should be added for that in NFA.



Set Construction Method:



ε-transitions

- NFA allows go to next state w/o any input
- ε means empty

Formal Notation for an ε-NFA

- **Definition:** an ε-NFA A is denoted by $A = (Q, \Sigma, \delta, q_0, F)$ where the transition function δ takes as arguments:
  - a state in Q, and
  - a member of $\Sigma \cup \{\varepsilon\}$

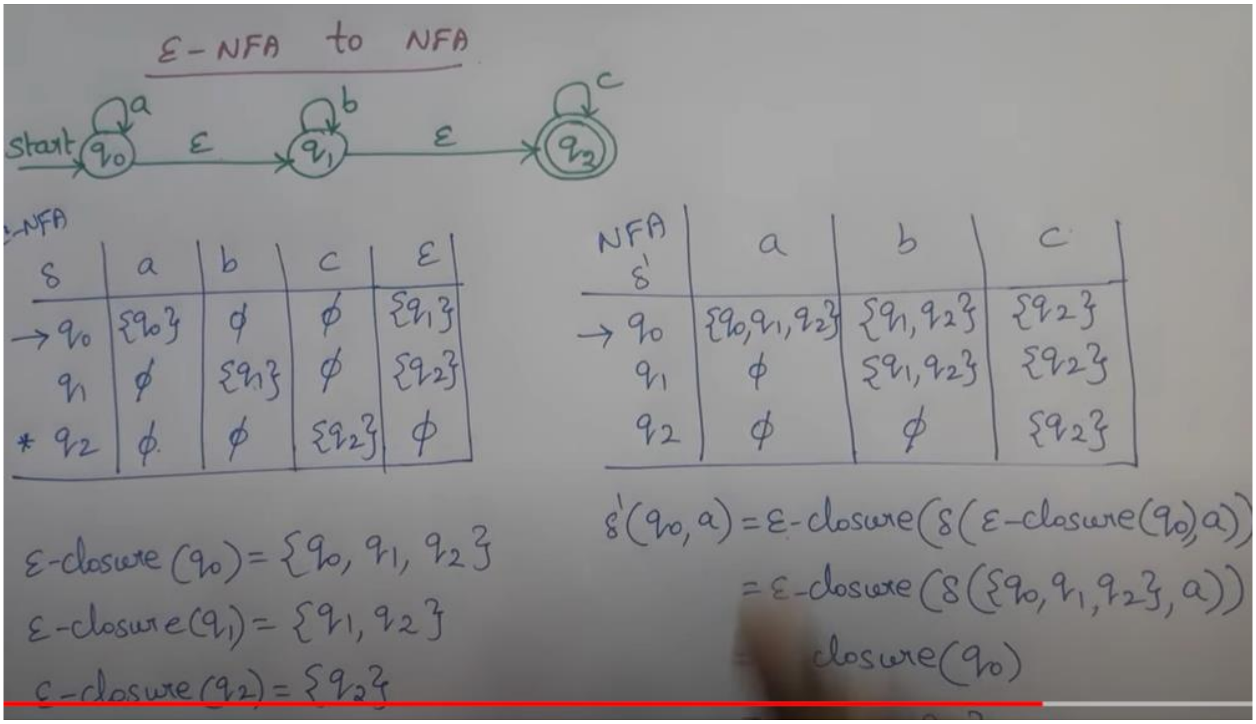
Epsilon-Closure

- Denoted by ε^*
- All the states than be reached from particular state only by seeing the ε symbol

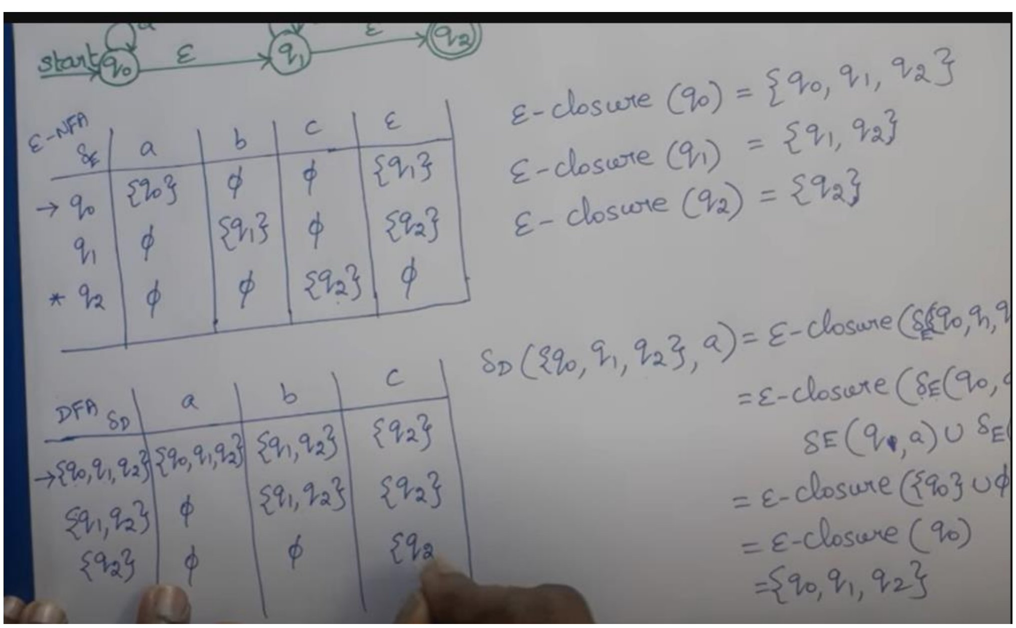
Epsilon-NFA to NFA:

- If a state (let x) goes to final state only by seeing ε in E-NFA, then the state (x) will be a final state in the NFA
- Using epsilon-closure

Epsilon-NFA to NFA

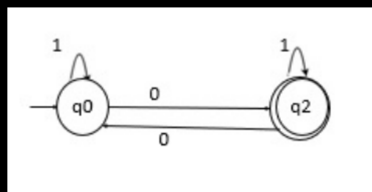


Epsilon NFA to DFA



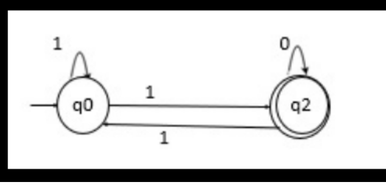
DFA accepting only odd number of 1's over an alphabet Σ={0,1}

The language L= {1,10,1110,1011,10101,100101,....}



DFA accepting only even number of 1's over an alphabet Σ={0,1}

The language L= {11,101,11110,10111,101101,1001011,....}



Now merge two transition diagrams to generate a machine accepting odd number of 0's or even number of 1's