Matrices, Vector Analysis and Co-ordinate Geometry

Created	@August 10, 2023 11:59 AM
Last edited time	@August 13, 2023 9:00 PM
Created by	B Borhan
Tags	CSTE Year 2 Term 1

1. **Define diagonal matrix, Scalar matrix, Hermitian matrix, Idempotent matrix and nil potent matrix**

Ans:

Diagonal Matrix : A square matrix in which every element except principle diagonal is zero.

Scalar Matrix : A type of diagonal matrix in which all diagonal element are same.

Hermitian matrix: A complex square matrix that is equal to its own conjugate transpose matrix.

Idempotent matrix: An idempotent matrix is one that when it multiplied by itself produces the same matrix.

Nilpotent matrix : A type of square matrix which produces a null matrix when it is multiplied by itself.

2. **Define diagonal and tri-diagonal matrix with examples.**

Ans:

Diagonal matrix: A square matrix in which every element except principle diagonal is zero.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

Tri-diagonal matrix: A square matrix in which every element except the major three diagonal is zero.

$$\begin{vmatrix} 1 & 6 & 0 \\ 4 & 2 & 7 \\ 0 & 5 & 3 \end{vmatrix}$$

3. Show that the matrices $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$ are the inverses of each other.

Ans:

$$|A| = 1(-8-3) + 2(2+4)$$
$$= -11 + 12 = 1$$

$$\text{adj}(A) = \begin{bmatrix} \begin{vmatrix} -1 & 3 \\ 1 & 8 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 1 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$
$$= \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$
$$= B$$

$$B^{-1} = \frac{1}{|B|} \text{adj}(B)$$
$$= A$$

$$|B| = -11(1) - 2(4-6) + 2(4)$$
$$= -11 + 4 + 8$$
$$= 1$$

$$\text{adj } B = \begin{vmatrix} \cdots\cdots \end{vmatrix}^T$$
$$= \begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{vmatrix}$$

5. Define Column matrix, Row matrix, Inverse matrix, Square matrix and Transpose of a matrix.

Ans:

Column matrix : A matrix having only 1 column

Row matrix : A matrix having only one row

Inverse matrix: If A is a non-singular matrix, there existence of $n * n$ matrix A^-1 which is called the inverse matrix of A such that it satisfies the property $AA^{-1} = I$, where I is an identity matrix

Square matrix: A matrix having same number of rows and columns

Transpose of a matrix : The transpose of a matrix can be defined as an operator which can switch the rows and column indices of a matrix.

6. Determine whether the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent or not.

Ans:

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & +2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, it's an idempotent matrix.

7. Find whether the matrix $A = \begin{bmatrix} 2 & 2-3i & 3+5i \\ 2+3i & 3 & i \\ 3-5i & -i & 5 \end{bmatrix}$ is Hermitian matrix or not.

$$A^T = \begin{bmatrix} 2 & 2+3i & 3-5i \\ 2-3i & 3 & -i \\ 3+5i & i & 5 \end{bmatrix}$$

$$\overline{A^T} = \begin{vmatrix} 2 & 2-3i & 3+5i \\ 2+3i & 3 & i \\ 3-5i & -i & 5 \end{vmatrix}$$

$$= A$$

So, it's Hermitian matrix

8. Solve the following equations for A and B

$$2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$$
$$2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

$$2(2A - B) = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix}$$
$$2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

$$\Rightarrow 5A = \begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix}$$
$$\cancel{5}A = \cancel{5}\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$(2B + A) - A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$
$$2B = \begin{bmatrix} 2 & 2 & 4 \\ -2 & 2 & -4 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

9. If the matrix $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is symmetric, find the value of x and hence find the matrix A.

$$A = A^T$$

$$\begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix} = \begin{bmatrix} 4 & 2x-3 \\ x+2 & x+1 \end{bmatrix}$$

$$x+2 = 2x-3$$

$$\Rightarrow \quad x = 5$$

12. Determine whether the following vectors are linearly dependent or linearly independent.
U = (1, 2, 5), v = (0, 2,4) and w = (-1, 1, 0).

> 💡 **determinant ≠ 0 → Linearly independent**
>
> **determinant = 0 → Linearly dependent**

$$X = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 2 & 4 \\ -1 & 1 & 0 \end{vmatrix}$$

$$|X| = -0 + 2(0+5) - 4(1+2)$$

$$= 10 - 12$$

$$= -2 \neq 0$$

Hence, it is **linearly independent.**

13. Determine the value of a so that the following system in unknowns x, y, z has:
(i) no solution (ii) more than one solution (iii) a unique solution.

$x + y - z = 1$

$2x + 3y + az = 3$

$x + ay + 3z = 2$

No Solutions:

For this to occur we must have a row where the left side is all zeros, but the right side is not

[0 0 0 | z]

Infinitely Many Solutions/More than one solution:

This occurs when we have a free variable. This is achieved by getting a whole row (including the right side) or column to be zeros:

[0 0 0 | 0]

Unique

We must have a leading term for each column.

$$D = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & a & 3 \\ 1 & a & 3 & 2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & a-1 & 4 & 1 \end{array}\right] \quad \left[\begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -(a+3) & 0 \\ 0 & 1 & a+2 & 1 \\ 0 & a-1 & 4 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -(a+3) & 0 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & -a^2-a+6 & 2-a \end{array}\right] \quad \left[R_3' = R_3 + R_2 - aR_2\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -(a+3) & 0 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & -(a+3)(a-2) & 2-a \end{array}\right]$$

$(i)\ a = -3$

$(ii)\ a = 2$

$(iii) a \neq -3, 2$

14. What is the rank of a matrix? Find the rank of the matrix $X = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 2 & 0 \\ -6 & 9 & -3 \end{bmatrix}$.

💡 **Rank of a Matrix**

The maximum number of linearly independent rows of a matrix is called the rank of a matrix.

Find the Row Echelon, count the number of non-zero row(s).

$$X = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 2 & 0 \\ -6 & 9 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & -4 & 4 \\ 0 & -9 & 9 \end{bmatrix} \begin{bmatrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 6R_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} R_2' = -\frac{R_2}{4} \\ R_3' = \frac{R_3}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\begin{matrix} 1 & -3 & 2 \\ 0 & 1 & -1 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} \end{bmatrix} \begin{bmatrix} R_3' = R_3 + R_2 \end{bmatrix}$$

The rank of the matrix is 2.

15. What is rank of a matrix? Reduce the following matrix A into its Echelon form to find the rank, where $A = \begin{bmatrix} 8 & 6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Rank of a matrix: The number of linearly independent rows in a matrix

$$A = \begin{bmatrix} 8 & 6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3/4 & 1/4 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \quad \begin{bmatrix} R_1 = R_1/8 \\ R_2 = R_2 + 6R_1 \\ R_3 = R_3 - 2R_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3/4 & 1/4 \\ 0 & 7 + \frac{9}{2} & -4 + \frac{3}{2} \\ 0 & -4 - \frac{3}{2} & 3 - \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3/4 & 1/4 \\ 0 & 23/2 & -5/2 \\ 0 & -11/2 & 5/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3/4 & 1/4 \\ 0 & 1 & -10/46 \\ 0 & -1 & 10/22 \end{bmatrix} \quad \begin{bmatrix} R_2' = \dfrac{R_2}{23/2} \\ R_3' = \dfrac{R_3}{11/2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3/4 & 1/4 \\ 0 & 1 & -10/46 \\ 0 & 0 & \frac{10}{22} - \frac{10}{46} \end{bmatrix} \quad \begin{bmatrix} R_3' = R_3 + R_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3/4 & 1/4 \\ 0 & 1 & -10/46 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} R_3' = \dfrac{R_3}{\frac{10}{22} - 10/46} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} C_2' = C_2 - \frac{3}{4} C_1 \\ C_3' = C_3 - \frac{1}{4} C_1 \\ C_3' = C_3 + \frac{10}{46} C_1 \end{bmatrix}$$

The rank of the matrix 3.

16. Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -3 & -4 \end{pmatrix} \begin{bmatrix} R_2' = R_2 - 4R_1 \\ R_3' = R_3 - 2R_1 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 2 \end{pmatrix} \begin{bmatrix} R_3' = R_3 - R_2 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} R_2' = \frac{R_2}{-3} \\ R_3' = \frac{R_3}{2} \end{bmatrix}$$

The rank of the matrix is 3.

17. Solve the following system of linear equations by

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$

i. using Cramer rule.
ii. using Gauss elimination.
iii. using matrix method

Which method involves fewer computations?

(i)

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 0 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$|D| = -2(-1-2) = 6$$

$$Dx = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} \qquad x = \frac{|Dx|}{|D|} = 1$$

$$|Dx| = 2(2+1) = 6$$

$$Dy = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 6 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & -1 \end{vmatrix} \qquad y = \frac{|Dy|}{|D|} = 2$$

$$|Dy| = 3(6-2) = 12$$

$$Dz = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} \quad |Dz| = 18 \qquad z = \frac{|Dz|}{D} = 3$$

(ii)

x

$$\left|\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 1 \end{array}\right|$$

$$= \left(\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & -1 & -3 & -11 \end{array}\right) \left[\begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3' - 2R_1 \end{array}\right]$$

$$= \left(\begin{array}{ccc:c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 3 & 11 \end{array}\right) \left[\begin{array}{l} R_2' = -\frac{R_2}{2} \\ R_3' = -R_3 \end{array}\right]$$

$$= \left(\begin{array}{ccc:c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right) \left[R_3' = R_3 - R_2 \right]$$

$$= \left(\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \left[\begin{array}{l} R_3' = \frac{1}{3} R_3 \\ R_1' = R_1 - R_3 \end{array}\right]$$

(iii)

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$B = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$|A| = 6 \neq 0$, unique solution exists.

$$A^{-1} = \frac{1}{6} \begin{vmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{vmatrix}^T$$

$$= \frac{1}{6} \begin{vmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{vmatrix}$$

$$AX = B$$

$$\Rightarrow X = A^{-1}B$$

$$= \frac{1}{6} \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 4+2 \\ 18-6 \\ 18+2-2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$2x + 3y + z = 11$

20. Find the characteristic equation and all the characteristic roots of the matrix $A = \begin{bmatrix} 8 & 6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 8 & 6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 8-\lambda & 6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (8-\lambda)\left[(7-\lambda)(3-\lambda) + 16\right] - 6\left[(-6)(3-\lambda) + 8\right] + 2\left[24 - 2(7-\lambda)\right]$$

$$= -\lambda^3 + 18\lambda^2 - 117\lambda + 120$$

$$|A - \lambda I| = 0$$

or, $\lambda^3 - 18\lambda^2 + 117\lambda - 120 = 0$

or $\lambda = 1.2, 8.3, 8.3 \ldots$

Question:

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Find eigenvectors ...

$|A - \lambda I| = 0$

$$\begin{vmatrix} (8-\lambda) & -6 & 2 \\ -6 & (7-\lambda) & -4 \\ 2 & -4 & (3-\lambda) \end{vmatrix} = 0$$

$(8 - \lambda)((7 - \lambda) \times (3 - \lambda) - (-4) \times (-4)) - (-6)((-6) \times (3 - \lambda) - (-4) \times 2) + 2((-6) \times (-4) - (7 - \lambda) \times 2) = 0$

$(8 - \lambda)\left(\left(21 - 10\lambda + \lambda^2\right) - 16\right) + 6((-18 + 6\lambda) - (-8)) + 2(24 - (14 - 2\lambda)) = 0$

$(8 - \lambda)\left(5 - 10\lambda + \lambda^2\right) + 6(-10 + 6\lambda) + 2(10 + 2\lambda) = 0$

$\left(40 - 85\lambda + 18\lambda^2 - \lambda^3\right) + (-60 + 36\lambda) + (20 + 4\lambda) = 0$

$\left(-\lambda^3 + 18\lambda^2 - 45\lambda\right) = 0$

$-\lambda(\lambda - 3)(\lambda - 15) = 0$

$\therefore$ the eigenvalues of the matrix A are given by $\lambda = 0, 3, 15$

21. Find the characteristic equation and all the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (1-\lambda)\left[(2-\lambda)^2\right] - 2\left[0-0\right] + 3\left[0-0\right]$$

$$= (1-\lambda)(2-\lambda)^2$$

Characteristic equation

$$(1-\lambda)(2-\lambda)^2 = 0$$

Root

$$(1-\lambda) = 0 \Rightarrow \lambda = 1$$

$$(2-\lambda)^2 = 0, \ \lambda = 2, 2$$

22. Diagonalize the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

💡 **Steps:**

1) Eigen Value (λ)
2) Eigen Vector (V_1, V_2)
3) Model Matrix $P = [V_1 \quad V_2]$
4) Diagonalized Matrix : $P^{-1}AP$

$$A - \lambda I = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda) - 8$$

$$= 3 - \lambda - 3\lambda + \lambda^2 - 8$$

$$= \lambda^2 - 4\lambda - 5$$

$$= (\lambda-5)(\lambda+1)$$

Characteristics equation

$$(\lambda-5)(\lambda+1) = 0$$

$$\Rightarrow \lambda = 5, -1$$

(i) $\lambda = -1$, $\boxed{(A - \lambda I) X}$

Eigen vector

$$[A+I]X = 0$$

$$\Rightarrow \begin{bmatrix} 1+1 & 4 \\ 2 & 3+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x + 4y = 0$$

$$\Rightarrow \frac{x}{2} = \frac{y}{-1} \qquad \bar{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(ii) $\lambda = 5$,

Eigen vector

$$[A - 5I]X = 0$$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4x + 4y = 0$$

$$\Rightarrow x = y = 1$$

$$\bar{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Model matrix:

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 5 & 10 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -3 & 0 \\ 0 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

23. State the Cayley Hamilton theorem. Verify the theorem for the matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and hence find A^{-1}.

💡 **Cayley Hamilton Theorem**

Cayley Hamilton theorem states that **every square matrix satisfies it's own equation.**

Verification

1) Figure out the characteristics equation
2) Replace A (matrix) instead of λ
3) Add an identity matrix with constant

$$A-\lambda I = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 3 \\ -1 & 4-\lambda \end{bmatrix}$$

$$|A-\lambda I| = (2-\lambda)(4-\lambda) + 3$$

$$= 11 - 6\lambda + \lambda^2$$

Characteristics equation

$$\lambda^2 - 6\lambda - 11 = 0$$

$$A^2 - 6A - 11I$$

$$\Rightarrow \begin{bmatrix} 1 & 18 \\ -6 & 13 \end{bmatrix} - \begin{bmatrix} 12 & 18 \\ -6 & 24 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$$

$$= 0 \text{ (verified)}$$

$$A^2 - 6A - 11I = 0$$

$$\Rightarrow AI - 6I - 11A^{-1} = 0$$

$$\Rightarrow 11A^{-1} = 6I - A$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}}$$

24. Verify Cayley Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$ and hence find the inverse of A.

Find the characteristic polynomial of the matrix M with respect to the variable λ:

$$M = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

To find the characteristic polynomial of a matrix, subtract a variable multiplied by the identity matrix and take the determinant:

$|M - \lambda \mathbb{I}|$

$$|M - \lambda \mathbb{I}| = \begin{vmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{vmatrix} - \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda + 1 & -3 & 3 \\ 3 & -\lambda - 5 & 3 \\ 6 & -6 & -\lambda + 4 \end{vmatrix}$$

Because there are no zeros in the matrix, expand with respect to row 1:

$$= \begin{vmatrix} -\lambda + 1 & -3 & 3 \\ 3 & -\lambda - 5 & 3 \\ 6 & -6 & -\lambda + 4 \end{vmatrix}$$

The determinant of the matrix $\begin{pmatrix} -\lambda + 1 & -3 & 3 \\ 3 & -\lambda - 5 & 3 \\ 6 & -6 & -\lambda + 4 \end{pmatrix}$ is given by

$(-\lambda + 1) \begin{vmatrix} -\lambda - 5 & 3 \\ -6 & -\lambda + 4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 3 \\ 6 & -\lambda + 4 \end{vmatrix} + 3 \begin{vmatrix} 3 & -\lambda - 5 \\ 6 & -6 \end{vmatrix}$:

$$= (-\lambda + 1) \begin{vmatrix} -\lambda - 5 & 3 \\ -6 & -\lambda + 4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 3 \\ 6 & -\lambda + 4 \end{vmatrix} + 3 \begin{vmatrix} 3 & -\lambda - 5 \\ 6 & -6 \end{vmatrix}$$

$(-\lambda + 1) \begin{vmatrix} -\lambda - 5 & 3 \\ -6 & -\lambda + 4 \end{vmatrix} = \left((-\lambda + 1)\left(\lambda^2 + \lambda - 2\right) = -\lambda^3 + 3\lambda - 2\right)$:

$$= \boxed{(-\lambda + 1)\left(\lambda^2 + \lambda - 2\right)} + 3 \begin{vmatrix} 3 & 3 \\ 6 & -\lambda + 4 \end{vmatrix} + 3 \begin{vmatrix} 3 & -\lambda - 5 \\ 6 & -6 \end{vmatrix}$$

$3 \begin{vmatrix} 3 & 3 \\ 6 & -\lambda + 4 \end{vmatrix} = (3(-3\lambda - 6) = -9\lambda - 18)$:

$$= (-\lambda + 1)\left(\lambda^2 + \lambda - 2\right) + \boxed{3(-3\lambda - 6)} + 3 \begin{vmatrix} 3 & -\lambda - 5 \\ 6 & -6 \end{vmatrix}$$

$3 \begin{vmatrix} 3 & -\lambda - 5 \\ 6 & -6 \end{vmatrix} = (3(6\lambda + 12) = 18\lambda + 36)$:

$$= (-\lambda + 1)\left(\lambda^2 + \lambda - 2\right) + 3(-3\lambda - 6) + \boxed{3(6\lambda + 12)}$$

$(-\lambda + 1)\left(\lambda^2 + \lambda - 2\right) + 3(-3\lambda - 6) + \boxed{3(6\lambda + 12)} = -\lambda^3 + 12\lambda + 16$:

Answer:

$$= -\lambda^3 + 12\lambda + 16$$

25. Show the following system of linear equations as its equivalent matrix form and as linear combination of vectors:

$$x_1 + 2x_2 - 4x_3 + 7x_4 = 4$$
$$3x_1 + x_2 + 6x_3 - 8x_4 = 12$$
$$4x_1 - 5x_2 - 3x_3 + 7x_4 = 8$$

Matrix representation

$$\begin{bmatrix} 1 & 2 & -4 & 7 \\ 3 & 1 & 6 & -8 \\ 4 & -5 & -3 & 7 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 8 \end{pmatrix}$$

Vector representation

$$x_1 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 6 \\ 3 \end{pmatrix} + x_4 \begin{pmatrix} -7 \\ 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 8 \end{pmatrix}$$

26. Give the matrix and vector representation of the following system of linear equations:

(i)
$$x_1 + 2x_2 - 4x_3 + 7x_4 = 4$$
$$3x_1 - 5x_2 + 6x_3 - 8x_4 = 8$$
$$4x_1 - 3x_2 - 2x_3 + 6x_4 = 11$$

(ii)
$$x_1 + 2x_2 = 40$$
$$-x_1 - 2x_2 + 6x_3 = 11$$
$$-5x_2 + 6x_3 - 8x_4 = 11$$
$$-2x_3 + 6x_4 - x_5 = 11$$
$$2x_4 - 7x_5 = 200$$

Matrix Representation

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ -1 & -2 & 6 & 0 & 0 \\ 0 & -5 & 6 & -8 & 0 \\ 0 & 0 & -2 & 6 & -1 \\ 0 & 0 & 0 & 2 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 40 \\ 11 \\ 11 \\ 11 \\ 200 \end{pmatrix}$$

Vector Representation

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -2 \\ -5 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 6 \\ 6 \\ -2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -8 \\ 6 \\ 2 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -7 \end{pmatrix} = \begin{pmatrix} 40 \\ 11 \\ 11 \\ 11 \\ 200 \end{pmatrix}$$

27. Define equal vector and null vector. Find the scalar product of the vectors (2, 3, 1) and (3, 1, -2). Also find the angle between them.

Equal Vector: Equal vectors are defined as **two vectors having same magnitude and direction**

Null Vector: A **directionless** vector whose **magnitude is zero** is called a null vector

u=2i + 3j+ 1 |u| = $\sqrt{2^2 + 3^2 + 1} = \sqrt{14}$

v = 3i + 1j -2k |v| = $\sqrt{14}$

u.v = (2*3) + (3*1) + (1)(-2) = 7

u.v = |u| . |v| cosθ.

$\theta = \cos^{-1} \frac{u.v}{|u|.|v|}$

$= \cos^{-1}\left(\frac{7}{14}\right)$

29. Apply the concept of vector cross product to find the area of the parallelogram constructed by the vectors $\vec{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

$\bar{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ $\bar{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$\text{Area} = \begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ 5 & 3 & 0 \end{vmatrix}$

$= -k(-6-15)$

$= 21k$

$|\text{Area}| = \sqrt{(21)^2}$

30. If $A = [a_{ij}]$, where $a_{ij} = \begin{cases} 0, & when\ i \neq j \\ C, & when\ i = j \end{cases}$

Construct a 3×3 order matrix and identify the type of matrix, where C is the sum of the 1st digit and the last digit of your ID. Also test the matrix A is

i. orthogonal or not

ii. singular or not

ID = 2101008

sum = 2 + 8 = 10

$\boxed{C = 10}$

$$A = \begin{vmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{vmatrix}$$

$$A^T = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$ → Square, diagonal, scalar matrix

$$A \cdot A^T = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

$$= 100 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 100I$$

$A^T A = 100I$

$A A^T = A^T A$

$AA^T = A^T A \neq I$, **not orthogonal**

$$A = \begin{vmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{vmatrix}$$

$$|A| = 10(100 - 0) - 0 + 0$$

$$= 1000 \neq 0$$, Not Singular

31. If $A = [a_{ij}]$ where $a_{ij} = \begin{cases} 0, & when\ i < j \\ i + j, & when\ i = j \\ 2i - j, & when\ i > j \end{cases}$

Construct a 3×3 matrix and identify the type of the matrix A. Also check whether it is singular or not.

Observation : i > j : below the principle diagonal
i < j : above the principle diagonal
i==j : the principle diagonal

$$A = \begin{vmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 4 & 6 \end{vmatrix}$$

A is square matrix, lower triangular matrix

$$|A| = 2(24 - 0) - 0 + 0$$
$$= 48 \neq 0,$$ Not singular

32. Define Augmented matrix. Use Augmented matrix to solve the system of linear equations

$2x + y - 2z = 10$
$3x + 2y + 2z = 1$
$5x + 4y + 3z = 4$

An augmented matrix is a matrix formed by combining the columns of two matrices to form a new matrix.

$$X = \left|\begin{array}{ccc:c} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{array}\right|$$

$$= \left|\begin{array}{ccc:c} 2 & 1 & -2 & 10 \\ 0 & 1 & 10 & -28 \\ 0 & 3 & 16 & -42 \end{array}\right| \left[\begin{array}{l} R_2' = 2 \times R_2 - 3R_1 \\ R_3' = 2 \times R_3 - 5R_1 \end{array}\right]$$

$$= \left|\begin{array}{ccc:c} 2 & 1 & -2 & 10 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & -14 & 42 \end{array}\right| \left[R_3' = R_3 - 3R_2 \right]$$

$$= \left|\begin{array}{ccc:c} 2 & 0 & -12 & 38 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & 1 & -3 \end{array}\right| \quad \left[\begin{array}{l} R_3' = -\frac{R_3}{14} \\ R_1' = R_1 - R_2 \end{array}\right]$$

$$= \left|\begin{array}{ccc:c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right| \quad \left[\begin{array}{l} R_1' = R_1 + 12R_3 \\ R_2' = R_2 - 10R_3 \end{array}\right]$$

$$= \left|\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right| \quad \left[R_1' = \frac{R_1}{2}\right]$$

33. Find a unit vector perpendicular to each of the vectors $\mathrm{r}_1 = 3\mathrm{i} + 2\mathrm{j} - 4\mathrm{k}$ and $\mathrm{r}_2 = \mathrm{i} + \mathrm{j} + 2\mathrm{k}$

$$R_1 \times R_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -4 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= i(4+4) - j(6+4) + k(3-2)$$

$$= 8i - 10j + k$$

$$|R_1 \times R_2| = \sqrt{8^2 + 10^2 + 1^2}$$

$$= \sqrt{167}$$

$$\hat{u} = \frac{R_1 \times R_2}{|R_1 \times R_2|}$$

$$= \frac{8}{\sqrt{167}} i - \frac{10}{\sqrt{167}} j + \frac{1}{\sqrt{167}} k$$

34. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

$$f(x,y,z) = x^2y + 2xz$$

$$\Delta f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(2xy+2z) + j(x^2) + k(2x)$$

$$\Delta f(2,-2,3) = -2i + 4j + 4k$$

$$|\Delta f| = \sqrt{2^2+4^2+4^2} = 6$$

$$a = \frac{\Delta f}{|\Delta f|} = \frac{1}{3}(-i+2j+2k)$$

35. Determine whether the force field $\vec{F}(x,y,z) = x^2y\hat{\imath} + xyz\hat{\jmath} - x^2y^2\hat{k}$ is a conservative or not.

💡 **Curl (F) = 0 → Conservative & Irrotational**

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xyz & x^2 y^2 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(x^2 y^2) - \frac{\partial}{\partial z}(xyz)\right]$$

$$- j\left[\frac{\partial}{\partial x} x^2 y^2 - \frac{\partial}{\partial z} x^2 y\right]$$

$$+ k\left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(x^2 y)\right]$$

$$= i\,(2yx - xy) - j\,(2xy^2 - 0)$$

$$+ k\,(yz - x^2)$$

$$\neq 0$$

36. Find the angle between the surfaces $x^2 + y^2 + z^2 = 49$ and $x^2 + y^2 - z = 43$ at (6, 3, -2).

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

$$\phi_1 = x^2 + y^2 + z^2 - 49$$

$$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$= 2xi + 2yj + 2zk$$

$$\nabla\phi_1 (6,3,-2),\quad \nabla\phi_1 = 12i + 6j - 4k$$

$$|\nabla\phi_1| = 14$$

$$\phi_2 = x^2 + y^2 - z - 13$$

$$\nabla\phi_2 = i(2x) + j(2y) - k$$

$$\nabla\phi_2(6,3,-2) = 12i + 6j - k$$

$$|\nabla\phi_2| = \sqrt{181}$$

$$\cos\theta = \frac{(12i+6j-9k)(12i+6j-k)}{14 \times \sqrt{181}}$$

$$= \frac{184}{14 \times \sqrt{181}}$$

$$\Rightarrow \theta = \cos^{-1}(\ldots) = 12.74$$

Final List

34. **Write down three vector operators gradient, divergence and curl.**

Gradient: The gradient of a function is defined to be a vector field. **It *denotes the direction of greatest change of a scalar function .***The gradient of a scalar-valued function $f(x, y, z)$ is the vector field

$grad\ f = \Delta f = \frac{\delta f}{\delta x}i + \frac{\delta f}{\delta y}j + \frac{\delta f}{\delta z}k$

Δf is a vector valued function, but f is not.

Divergence: Divergence is a **vector operator** that operates on a **vector field**, producing a scalar field giving the **quantity of the vector field's source at each point**. The divergence of a vector field $F(x, y, z)$ is the scalar-valued function

$divF = \Delta .F = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}$

F is vector-valued function but, $divF$ is not.

Curl: Curl is a **vector operator** that describes the **infinitesimal circulation** of a vector field in **three-dimensional** Euclidean space. The curl of a vector field $F(x, y, z)$ is the vector field

$curl\ F = \Delta \times F$

F and $curl\ F$are both vector-valued function.

35. **Show that the divergence of the curl of a vector field A is zero.**

$$\bar{F} = F_x i + F_y j + F_z k$$

$$\text{curl}(\bar{F}) = \nabla \times \bar{F}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= i\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - j\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)$$
$$+ k\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\text{div}(\text{curl}\,\bar{F}) = \nabla \cdot (\nabla \times \bar{F})$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)$$
$$+ \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= 0$$

36. Let $\vec{A} = xy^2\,\underline{i} - 3x^2y\,\underline{j} + 2yz^2\,\underline{k}$. Now find curlcurl of $\vec{A}$ at (1, 0, - 4).

$$\bar{A} = xy^2 i - 3x^2y j + 2yz^2 k$$

$$\text{curl}(\bar{A}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & -3x^2y & 2yz^2 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(2yz^2) - \frac{\partial}{\partial z}(-3x^2y)\right)$$

$$- j\left(\frac{\partial}{\partial x}(2yz^2) - \frac{\partial}{\partial z}(xy^2)\right)$$

$$+ k\left[\frac{\partial}{\partial x}(-3x^2y) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= i(2z^2) - j(0) + k(-6xy - 2xy)$$

$$\text{curl}(\text{curl}\,\bar{A}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z^2 & 0 & 4xy \end{vmatrix}$$

$$= i(4x - 0) - j(4y - 4z)$$

$$+ k(0 - 0)$$

$$(1, 0, -4) = i.(4) - j(4 \times 0 - 4(-4))$$

$$= 4i + 16j$$

37. Verify Green's theorem in the plane for $\oint_c (xy + y^2)dx + x^2dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Given,

$$\oint_c (xy+y^2)dx + x^2dy$$

$M = xy + y^2 \qquad N = x^2$

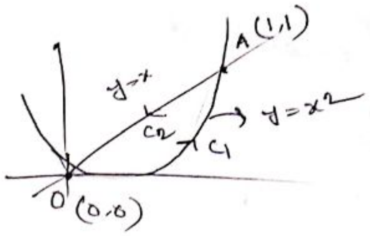
$\Rightarrow \frac{\partial M}{\partial y} = x + 2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$

$y = x, \quad y = x^2$

$\Rightarrow x^2 = x$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0, 1. \quad y = 0, 1$



$$\text{L.H.S} = \oint (xy+y^2)\,dx + x^2\,dy$$

$$= \int_{OA} (xy+y^2)\,dx + x^2\,dy + \int_{AO} (xy+y^2)\,dx + x^2\,dy$$

$$= I_1 + I_2$$

I_1.

Along OA,

$y = x^2$

$\Rightarrow dy = 2x\,dx$

$x = 0$ to 1

$I_1 = \int_{OA} (xy+y^2)dx + x^2dy$

$= \int_0^1 (x^3+x^4)dx + x^2 \cdot 2x \cdot dx$

$= \frac{1}{4} + \frac{1}{5} + \cancel{2} \cdot \frac{1}{\cancel{4}\,2}$

$= \frac{1}{4} + \frac{1}{5} + \frac{1}{2} = \frac{19}{20}$

Along AO,

$y = x$

$dy = dx$

$x = \cancel{0 \text{ to } 1}$ 1 to 0

$I_2 = \int_1^0 (x^2+x^2)dx + x^2dx$

$= -3 \cdot \frac{1}{3}$

$= -1$

$$\text{L.H.S} = \frac{19}{20} - 1 = \frac{-1}{20}$$

$$\text{R.H.S} = \int_{x=0}^{1} \int_{y=x^2}^{x} (2x - x - 2y)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx = \int_0^1 \left[xy - 2\frac{y^2}{2}\right]_{x^2}^{x} dx = \int_0^1 \left((x^2 - x^2) - (x^3 - x^4)\right) dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{\ }$$ [illegible]

39. Apply Green's theorem find $\oint_c (x^2 y dx + x^2 dy)$, where c is the boundary of the region enclosed by the line y = x and the curve $y = x^2$

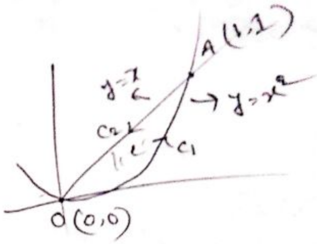
(39) Given, $\oint (x^2 y\,dx + x^2 dy)$

$M = x^2 y$ $\quad N = x^2$

$\Rightarrow \frac{\partial M}{\partial y} = x^2 \quad \Rightarrow \frac{\partial N}{\partial x} = 2x$

$y = x, \quad y = x^2$

$\Rightarrow x = 0, 1, \quad y = 0, 1$



From Green's theorem,

$$c = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (2x - x^2)\,dy\,dx$$

$$= \int_0^1 \left[2xy - x^2 y\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[(2x^2 - x^3) - (2x^3 - x^4)\right] dx$$

$$= \int_0^1 (2x^2 - 3x^3 + x^4)\,dx$$

$$= 2 \cdot \frac{1}{3} - 3 \cdot \frac{1}{4} + \frac{1}{5}$$

$$= \frac{40 - 45 + 12}{3 \times 4 \times 5} = \frac{7}{60}$$

40. Determine the angles α, β, γ which the vector $\vec{A} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ makes with the positive directions of the coordinate axes. Also show that $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$.

$\bar{A} = 2i - 3j + k \quad |A| = 14$

$\alpha = \cos^{-1} \frac{(2i-3j+k)\cdot i}{|\bar{A}|\cdot|i|}$

$= \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$

$\beta = \cos^{-1} \frac{\bar{A}\cdot j}{|A||j|} = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$

$\gamma = \cos^{-1} \frac{\bar{A}\cdot k}{|A|\cdot|k|} = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$

$L.H.S = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$= 1$

41. Find a unit vector
   i. in the direction to the vector $\vec{A} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$.
   ii. parallel to the resultant of the vectors $\vec{B} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$ and $\vec{C} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.
   iii. perpendicular to the plane constructed by the vectors $\vec{D} = 3\hat{\imath} + \hat{\jmath}$ and $\vec{E} = -\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

(i) $\hat{u} = \frac{A}{|A|} \quad |A| = 3\sqrt{5}$

$= \frac{2i + 4j - 5k}{3\sqrt{5}}$

(ii) $\bar{A} + \bar{B} = 3i + 6j - 2k$

$|\bar{A} + \bar{B}| = 7$

$\hat{u} = \frac{\bar{A} + \bar{B}}{|\bar{A} + \bar{B}|} = \boxed{}$

(iii) $\bar{D} \times \bar{E} = \begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ -1 & 2 & 2 \end{vmatrix}$

$= 2i - 6j + 7k$

$|\bar{D} \times \bar{E}| = \sqrt{89}$

$\hat{u} = \frac{\bar{D} \times \bar{E}}{|\bar{D} \times \bar{E}|} = \boxed{}$

42. Find whether the vectors $\vec{A} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$, $\vec{B} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{C} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$ are coplanar.

$$\bar{A} \times \bar{B} = \begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= i - 8j - 5k$$

$$(\bar{A} \times \bar{B}) \cdot \bar{C} = (i - 8j - 5k) \cdot (3i + j - k)$$

$$= 3 - 8 + 5$$

$= 0$, Coplanar

43. What is inner product of vectors? Apply the Gram-Schmidt orthonormalization algorithm to the set of vectors $v_1 = (1, 0, 1)$, $v_2 = (1, 0, -1)$ and $v_3 = (0, 3, 4)$ to obtain an orthonormal basis. Justify your results.

An inner product is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar.

$$U_1 = \frac{V_1}{||V_1||} = \frac{[1, 0, 1]}{\sqrt{1^2 + 0^2 + 1^2}} = \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$$

Then normalize $V_2 = [1, 0, -1]$

$$U_2 = \frac{V_2 - (V_2.U_1)U_1}{\|V_2 - (V_2.U_1)U_1\|}$$

$$V_2 - (V_2.U_1)U_1 = [1, 0, -1] - \left([1, 0, -1].\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]\right)\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$$

$$= [1, 0, -1] - \left(1.\frac{1}{\sqrt{2}} + 0.0 + (-1).\frac{1}{\sqrt{2}}\right)\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$$

$$= [1, 0, -1] - \left(1.\frac{1}{\sqrt{2}} + 0.0 + (-1).\frac{1}{\sqrt{2}}\right)\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$$

$$= [1, 0, -1]$$

$$\therefore\ U_2 = \frac{V_2 - (V_2.U_1)U_1}{\|V_2 - (V_2.U_1)U_1\|} = \frac{[1, 0, -1]}{\sqrt{1^2 + 0^2 + (-1)^2}} = \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]$$

Now normalize $V_3 = [0, 3, 4]$

$$U_3 = \frac{V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2}{\|V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2\|}$$

$$\begin{aligned} V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2 \\ &= [0, 3, 4] - \left([0, 3, 4].\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]\right)\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right] \\ &\quad - \left([0, 3, 4].\left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right]\right)\left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right] \\ &= [0, 3, 4] - [2, 0, 2] - [2, 0, -2] \\ &= [0, 3, 0] \end{aligned}$$

$$\therefore\ U_3 = \frac{V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2}{\|V_3 - (V_3.U_1)U_1 - (V_3.U_2)U_2\|} = \frac{[0, 3, 0]}{\sqrt{0^2 + 3^2 + 0^2}} = [0, 1, 0$$

Therefore, the orthonormal basis is $\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right], [0, 1, 0]$

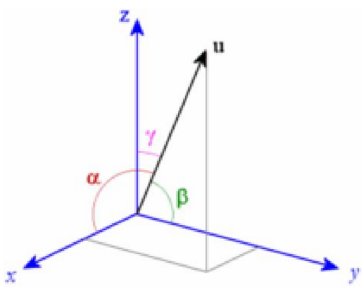
44. Explain direction cosines of a line. If the angle between two straight lines is θ and their direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 then show that

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

Hence develop this relation for $\sin\theta$.

Direction cosines of a line

If a given line makes angles α, β, γ with the positive direction of x, y and z axes respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of the line and are generally denoted by l, m, n respectively. The angles α, β, γ are called the direction angle of the line.



Second method : Let $OP_1 = r_1$, $OP_2 = r_2$, $P_1P_2 = d$ and the co-ordinates of P_1, P_2 be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

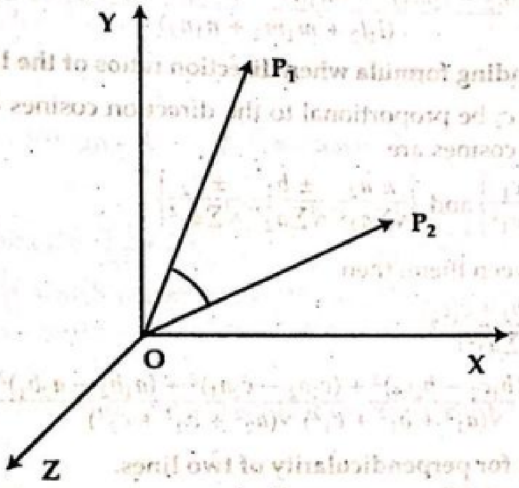


Fig: 11

Then $x_1 = r_1 l_1$; $y_1 = r_1 m_1$; $z_2 = r_1 n_1$ ... ... **(n)**

and $x_2 = r_2 l_2$; $y_2 = r_2 m_2$; $z_2 = r_2 n_2$

then by geometry, we have

$OP_1^2 + OP_2^2 - P_1P_2^2 = 2\, OP_1\, OP_2 \cos\theta$

or, $\cos\theta = \dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

Now $r_1^2 = x_1^2 + y_1^2 + z_1^2$, $r_2^2 = x_2^2 + y_2^2 + z_2^2$

and $P_1P_2^2 = d_2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$= r_1^2 + r_2^2 - 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2)$ by eq. (3)

$\therefore \cos\theta = \dfrac{r_1^2 + r_2^2 - r^2{}_1 - r_2^2 + 2r_1r_2\,(l_1l_2 + m_1m_2 + n_1n_2)}{2r_1r_2}$

or, $\cos\theta = l_1l_2 + m_1m + n_1n_2$ ... ... **(18)**

Art.17. (a) Expression for $\sin\theta$ and $\tan\theta$.

$\sin^2\theta = 1 - \cos^2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (l^2{}_1 + m_1^2 + n_1^2)\,(l^2{}_2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1\,n_2)^2$

$= (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2$ ... ... **(19)**

2. Explain shortest distance. Find the equation of the line of shortest distance and evaluate the length of the shortest distance between the lines
$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-5}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$

Shortest Distance

When two lines do not intersect and are parallel as well, that is, they do not lie in the same plane, then these lines are said to be non-intersecting lines. The straight line which is perpendicular to each of these non-intersecting lines is called the line of shortest distance and the length of this line intercepted between the given lines is called the shortest distance of those lines.

Length of the shortest distance

Let l, m, n be the direction cosines of the line of shortest distance. As it is perpendicular to both the lines given

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-5}{4}$$
$$\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

We get $2l+3m+4n=0$ and $3l+4m+5n=0$

Solving simultaneously we get

$$\frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}$$

Giving $\frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}=\frac{1}{\sqrt{6}}$, thus $l=-\frac{1}{\sqrt{6}}$, $m=\frac{2}{\sqrt{6}}$ and $n=-\frac{1}{\sqrt{6}}$

The magnitude of the shortest distance is the projection of the line joining $(1,2,5)$ and $(2,4,5)$

$\therefore$ Shortest Distance $=(2-1)\left(-\frac{1}{\sqrt{6}}\right)+(4-2)\left(\frac{2}{\sqrt{6}}\right)+(5-5)\left(-\frac{1}{\sqrt{6}}\right)=\frac{3}{\sqrt{6}}$

Now, the equation of the plane containing the first of the two given lines and the line of shortest distance is

$$\begin{vmatrix} x-1 & y-2 & z-5 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix}=0$$
$$11x+2y-7z+13=0$$

Also the equation of the plane containing the second line and the shortest distance is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix}=0$$
$$7x+y-5z+7=0$$

Therefore, the equation of the line of shortest distance is

3. Show that the lines $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$ are coplanar. Find their intersection point and the equation of the plane in which they lie.

The condition for the lines

$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ to be coplanar is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Now

$$\begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}=0$$

So the given lines are coplanar.

The equation of the plane in which they lie is

$$\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}=0$$

$$21x-19y+22z+125=0$$

$$\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}=a \qquad \frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}=b$$

$x=4a+5$ $\qquad$ $x=7b+8$

$y=4a+7$ $\qquad$ $y=b+4$

$z=-5a-3$ $\qquad$ $z=3b+5$

$$4a + 5 = 7b + 8$$
$$4a + 7 = b + 4$$
$$(-)$$

$$b = -1$$

$$x = 1$$
$$y = 3$$
$$z = 2$$

intersecting point $(1, 3, 2)$

47. Find the equation of the straight line that intersect the lines $4x + y - 10 = 0 = y + 2z + 6$ and $3x - 4y + 5z + 5 = 0 = x + 2y - 4z + 7$ and passing through the point $(-1, 2, 2)$.

$$4x + y - 10 = 0$$
$$y + 2z + 6 = 0$$

$$4x + y - 10 + k(y + 2z + 6) = 0$$

at $(-1, 2, 2)$

$$k = \frac{12}{12} = 1$$

$$\boxed{4x + 2y + 2z - 4 = 0}$$