Ordinary & Partial Differentiation

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Integration Formulae and tricks

01. $\int \sin x dx = -\cos x + c$

02. $\int \cos x dx = \sin x + c$

03. $\int \tan x dx = \ln(\sec x) + c$

04. $\int \operatorname{cosec} x \, dx = \ln(\operatorname{cosec} x - \cot x) + c = -\ln(\operatorname{cosec} x + \cot x) + c = \ln\left(\tan\frac{x}{2}\right) + c$

05. $\int \sec x \, dx = \ln(\sec x + \tan x) + c = \ln\left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\} + c = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + c$

06. $\int \cot x \, dx = \ln(\sin x) + c = -\ln(\operatorname{cosec} x) + c$

07. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \ (n \neq -1)$

08. $\int \sec^2 x \, dx = \tan x + c$

09. $\int \operatorname{cosec}^2 x dx = -\cot x + c$

10. $\int \sec x \tan x \, dx = \sec x + c$

11. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

12. $\int e^{mx} dx = \frac{1}{m} e^{mx} + c$

13. $\int a^x dx = \frac{a^x}{\ln a} + c$

14. $\int \frac{1}{x} dx = \ln x + c$

15. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ [$x = a\tan\theta$ ধরি]

16. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

17. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ [$x = a\sin\theta$ ধরি]

19. $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(\sqrt{x^2+a^2} + x\right) + c$ [$x = a\tan\theta$ ধরি]

20. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(\sqrt{x^2-a^2} + x\right) + c$ [$x = a\sec\theta$ ধরি]

21. $\int e^{mx} \{mf(x) + f'(x)\} dx = e^{mx} f(x) + c$

22. $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

23. $\int \frac{dz}{\sqrt{z}} = 2\sqrt{z} + c$

24. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

25. $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c$ $[x = a\tan\theta$ ধরি$]$

26. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

27. $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + c$ $[x = a\sec\theta$ ধরি$]$

28. $\int \ln x\,dx = x\ln x - x + c$

29. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

30. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

31. $\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$

Credit: **Udvash** Engineering Concept Book

All Formulae:
https://elearn.daffodilvarsity.edu.bd/pluginfile.php/578401/mod_resource/content/4/Formula of Differentiation and integration.pdf

Definition of Differential :

- involving derivatives of one or more dependent variable w.r.t one or more independent variable

Dependent/Independent Variable:

$$\frac{dy}{dx}$$

x = independent

y= dependent *(y depends on x, y is a function of x)*

Types :

- Ordinary Diff. : If independent variable is one
- Partial Diff : If independent variables are > one

Order:

- Highest order derivative

i. $\frac{dy}{dx} + 1 = 0 \rightarrow 1 \checkmark$

ii. $\frac{dy}{dx} = 3 \rightarrow 1 \checkmark$

iii. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \rightarrow 2 \checkmark$

iv. $(\underbrace{\frac{d^3y}{dx^3}}_{3})^2 + \frac{d^4y}{dx^4} + 5x = 2y \rightarrow 4 \checkmark$

<u>**Degree:**</u>

- Highest order derivative's power
- Degree can't be fractional, so if there's any fractional variable or derivatives **(not only the highest derivative, it is applicable for all variables)** we need to make it integer first.

  → How ? → Squaring, cubing…^n-ing the whole equation

Problems:

i. $\frac{dy}{dx} + 1 = 0$ — 1 ✓

ii. $\frac{dy}{dx} = 3$ — 1 ✓

iii. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ — 1 ✓

iv. $(\frac{d^3y}{dx^3})^2 + \frac{d^4y}{dx^4} + 5x = 2y$ — 1 ✓

- $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = y\frac{d^2y}{dx^2}$

order → 2

degree → $1+\left(\frac{dy}{dx}\right)^2 = y^2\left(\frac{d^2y}{dx^2}\right)^{2}$

- $\left(\frac{d^2y}{dx^2}\right)^{4/5} = -\left(\frac{dy}{dx}\right)^{5/4}$

$\Rightarrow \left(\frac{d^{\boxed{2}}y}{dx^2}\right)^{\boxed{16}} = (-1)^{25}\left(\frac{dy}{dx}\right)^{25}$

O → 2 D = 16

NB: Don't define degree if $\frac{1}{y}\frac{dy}{dx}$, $\frac{\frac{dy^2}{dx^2}}{\frac{dy}{dx}}$, first make it like multiple of y, $\frac{dy}{dx}$ for the example.

Transcendental Function:

Definition: In mathematics, when a function is not expressible in terms of a finite combination of algebraic operation of addition, subtraction, division, or multiplication raising to a power and extracting a root…

Keywords: function which are return a infinite series

- If **any derivative** locates like this sin(derivative function), cos(derivative function), e^(derivative function), then the degree will be undefined.

Example:

1) $e^{\frac{dy}{dx}} = 5y$

Order : 1

Degree : Undefined

2) $sin(\frac{dy}{dx})\frac{d^2y}{dx^2} + 6y = 0$

Order : 2

Degree : Not defined

3) $siny + \frac{d^3y}{dx^3} = 22$

Order : 3

Degree : 1 (No derivative function in the $sine$)

Wronskian Theorem:

Let $y_1(x)$ and $y_2(x)$ are two solution of second order linear differential equation then Wronskian of y_1, y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y'_1y_2$$

$W(y_1, y_2) == 0$ → Linearly Dependent Solution

$W(y_1, y_2) \neq 0$ → Linearly Independent Solution

If there're $y_1, ..., y_n$, then the equation will be like below

$$W(y_1, ..., y_n) = \begin{vmatrix} y_1 & y_2 & ... & y_n \\ y'_1 & y'_2 & ... & y'_n \\ ..... & & & \\ y^{n'}{}_1 & y^{n'}{}_2 & ... & y^{n'}{}_n \end{vmatrix}$$

Linear Differential Equation:

- $(\frac{dy^n}{d^n x})^m$ m must be 1
- There will be no $y.\frac{dy}{dx}$, $\frac{dy}{dx}\frac{dy^2}{d^2x}$, $\frac{dy}{dx}\frac{dz}{dx}$

Variable Separable Method:

Steps:

- Make the equation like that f(x) = f(y)
- Integrate it

Problems:

$$(x+1)\frac{dy}{dx} = x(y^2+1)$$

$$\Rightarrow (x+1)\,dy = dx \cdot x \cdot (y^2+1)$$

$$\Rightarrow \frac{dy}{y^2+1} = \frac{dx\,x}{x+1}$$

$$\Rightarrow \int \frac{dy}{y^2+1} = \int \left(1 - \frac{1}{x+1}\right) dx$$

$$\Rightarrow \tan^{-1} y = x - \ln(x+1) + C$$

$$* \; (xy^2 + x)dx + (yx^2 + y)dy = 0$$

$$\Rightarrow x\,dx\,(y^2+1) + y\,dy\,(x^2+1) = 0$$

$$\Rightarrow \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{2y}{y^2+1}\,dx = 0$$

$$\Rightarrow \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\ln(y^2+1) + C = 0$$

$$\Rightarrow \ln(x^2+1) + \ln(y^2+1) + C = 0$$

Reducible to Variable Separable Method:

If the equation like that,

$$\frac{dy}{dx} = f(ax + by + c)...(i)$$

C may be 0.

$ax + by + c = v$ or *something else to solve the problem*

$\frac{dy}{dx} = \frac{dv}{dx} + ...$ .. (ii)

Put $\frac{dy}{dx}$ from (ii) value in (i) and $ax + by + c = v$

Trick: term which one is repeated more than one = v

Problems**:**

- $\frac{dy}{dx} = (4x+y+1)^2$

$\boxed{\begin{aligned} &4x+y+1 = v \\ &\Rightarrow 4 + \frac{dy}{dx} = \frac{dv}{dx} \\ &\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4 \end{aligned}}$

$\Rightarrow \frac{dv}{dx} - 4 = v^2$

$\Rightarrow \int \frac{dv}{v^2+(2)^2} = \int dx$

$\Rightarrow \frac{1}{2} \tan^{-1}\left(\frac{v}{2}\right) = x$

$\Rightarrow \frac{1}{2} \tan\left(\frac{4x+y+1}{2}\right) = x$

- $x: \frac{dy}{dx} + x^3 y = -\sec(xy)$

$\Rightarrow x^3\left(\frac{dy}{dx}x + y\right) = -\sec(xy)$

$\boxed{\begin{aligned} &xy = v \\ &\Rightarrow \frac{dy}{dx} + y = \frac{dv}{dx} \end{aligned}}$

$\Rightarrow x^3 \frac{dv}{dx} = -\sec v$

$\Rightarrow \frac{dv}{\sec v} = -\frac{dx}{x^3}$

$\Rightarrow \int \cos v \, dv = -\int x^{-3} dx$

$\Rightarrow \sin v = +\frac{x^{-2}}{+2} + C$

$\Rightarrow \sin(xy) = \frac{1}{2x^2} + C$

$$\bullet\ y\sqrt{x^2-1}\,dx + x\sqrt{y^2-1}\,dy = 0$$

$$\Rightarrow \frac{\sqrt{x^2-1}}{x}\,dx = -\frac{\sqrt{y^2-1}}{y}\,dy$$

$$\Rightarrow \int \frac{u^2}{u^2+1}\,du = -\int \frac{w^2}{w^2+1}\,dw$$

$$\Rightarrow \int \left(1-\frac{1}{u^2+1}\right)du = -\int \left(1-\frac{1}{w^2+1}\right)dw$$

$$\Rightarrow u - \tan^{-1}(u) = -w + \tan^{-1}(w)$$

$$\Rightarrow \sqrt{x^2-1} - \tan^{-1}\left(\sqrt{x^2-1}\right) = -\sqrt{y^2-1} + \tan^{-1}\sqrt{y^2-1}$$

$$\sqrt{x^2-1} = u$$

$$\Rightarrow x^2-1 = u^2$$

$$\Rightarrow x^2 = u^2+1$$

$$\Rightarrow 2x = 2u\frac{du}{dx}$$

$$\Rightarrow dx = du$$

$$\Rightarrow x\,dx =$$

2019-20 → 2(a)

Mistake : $xdx = udu$

Linear Differentiation Equation:

Definition: If P and Q are only functions of x or constants then the differential equation of the form $dy/dx + Py = Q$ is called the first order linear differential equation.

$$\frac{dy}{dx} + yP = Q$$

Where P and Q are constant or function of x. Then, linear differentiable equation is applicable. Otherwise Separable Method.

Step:

- Integrating Factor $I_f = e^{\int Pdx}$

A given differential equation may not be integrable as such. But it may become integrable when it
is multiplied by a function. Such a function is called the integrating factor (I.F).

dx = independent variable. If y is an independent variable then, it would be **Pdy.**

- General Solution: and Calculate

$$yI_f = \int QI_f dx + C$$

Problems:

- $$\frac{dy}{dx} + 4y = x$$

$$P = 4 \qquad Q = x$$

$$\text{(1)} \quad IF = e^{\int P\,dx} = e^{4x}$$

$$\text{(2)} \quad y \cdot e^{\int P\,dx} = \int x \cdot e^{\int P\,dx} \cdot dx + C$$

$$\Rightarrow y\,e^{4x} = x \int e^{4x}\,dx - \int \frac{dx}{dy} \int e^{4x}\,dx \cdot dx + c$$

$$= x \cdot \frac{e^{4x}}{4} - \int \frac{e^{4x}}{4}\,dx + C$$

$$= x \cdot \frac{e^{4x}}{4} - e^{4x}/16 + C$$

$$\Rightarrow \boxed{y = x/4 - 1/16 + C}$$

$\frac{dy}{dx} + y = \cos x$

(1) $IF = e^{\int 1 dx} = e^{x}$

(2) $y \cdot e^{x} = \int e^{x} \cdot \cos x \cdot dx$

$I = \int e^{x} \cdot \cos x \, dx$

$= \cos x \cdot e^{x} + \int \sin x \int e^{x} dx \, dx$

$= \cos x \cdot e^{x} + \int \sin x \, e^{x} dx$

$= \cos x \cdot e^{x} + e^{x} \cdot \sin x - \underbrace{\int \cos x \, e^{x} dx}_{I}$

$2I = \cos x \cdot e^{x} + e^{x} \cdot \sin x$

$\Rightarrow I = \frac{e^{x}(\cos x + \sin x)}{2}$

$$\boxed{y \cdot e^{x} = \frac{e^{x}(\cos x + \sin x)}{2} + C}$$

Homogenous Differentiation Equation:

Definition: An equation of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ in which $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of x and y of the same degree can be reduced to an equation in which variables are separable by putting $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Homogenous Function & It's Degree: $F(x,y)$ is homogenous function and n is the degree of the function then if $f(mx, my) = m^n f(x,y)$, where m is a non-zero value.

$$f(x,y)=\frac{x^2+y^2}{x+y}$$

$$\Rightarrow f(mx,my)=\frac{m^2x^2+m^2y^2}{mx+my}$$

$$= m^{2-1}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= m^{1} f(x,y)$$

Step to Solve a Homogenous Differentiation Equation:

- Form the equation like $\frac{dy}{dx} = ...$
- $\frac{dy}{dx} = \frac{f(x,y)}{f(x,y)}$ Check the homogenous of the functions and degree individually, the functions must be homogenous and their degree must be same
- Put $y = vx$ int Step (1) equation and $\frac{dy}{dx} = v + x\frac{dv}{dx}$
- Apply Variable Separable Method
- Put $v = \frac{y}{x}$

Problems:

$\circ\ x^2y\,dx - (x^3+y^3)\,dy = 0$

1) $\dfrac{dy}{dx} = \dfrac{x^2y}{x^3+y^3}$ $\to$ 3rd $\to$ 3rd ✓

3) $y = vx,$

$$v + x\frac{dv}{dx} = \frac{x^3v}{x^3+v^3x^3}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-v-v^4}{1+v^3}$$

4)

$$\Rightarrow \int\frac{dx}{x} = -\int\frac{1+v^3}{v^4}\,dv$$

$$\Rightarrow \ln x = -\int\frac{1}{v^4}\,dv - \int\frac{1}{v}\,dv$$

$$\Rightarrow \ln x = -\frac{v^{-3}}{-3} - \ln v + C$$

5)

$$\Rightarrow \ln x = +\frac{1}{3}\left(\frac{y}{x}\right)^{-3} - \ln\left(\frac{y}{x}\right) + C$$

$$\Rightarrow \ln\left(\frac{xy}{x}\right) = \frac{1}{3}\left(\frac{y}{x}\right)^{-3} + C$$

$$\Rightarrow \boxed{\ln(y) = \frac{1}{3}\left(y/x\right)^{-3} + C}$$

$\circ \quad \frac{dy}{dx} = y/x + \sec(y/x)$

$y = vx$

$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + \sec(v)$

$\Rightarrow \quad \int \frac{dx}{x} = \int \cos(v)\, dv$

$\Rightarrow \quad \ln x = \sin v + \ln c$

$\Rightarrow \quad \ln x = \ln e^{\sin v} + \ln c$

$\Rightarrow \quad x = c e^{\sin(y/x)}$

Exact Differentiation Equation:

Definition: $M(x,y)dx + N(x,y)dy = 0$ the equation will be Exact equation if $\frac{dM(x,y)}{dy} = \frac{dN(x,y)}{dx}$

[NB: y is constant when diff.w.r.t x and x is constant when diff.w.r.t y]

Steps:

- Check it a exact differentiation or not
- General Solution:

  $\int M(x,y)dx \; [y \; is \; constant] + \int N(x,y)dy \; [terms \; free \; from \; x] = C$

Problems:

○ $x(x^2+y^2-a)dx + y(x^2-y^2-a)dy$

(1) $x^3 - y^2x - ax = M(x,y)$

$x^2y - y^3 - ay = N(x,y)$

$\left.\begin{array}{l}\dfrac{\partial M(x,y)}{\partial y} = 2yx \\ \dfrac{\partial N(x,y)}{\partial x} = 2xy\end{array}\right\}$ EDE

(2) $\int (x^3 - y^2x - ax)\,dx + \int (-y^3 - ay)\,dy = C$

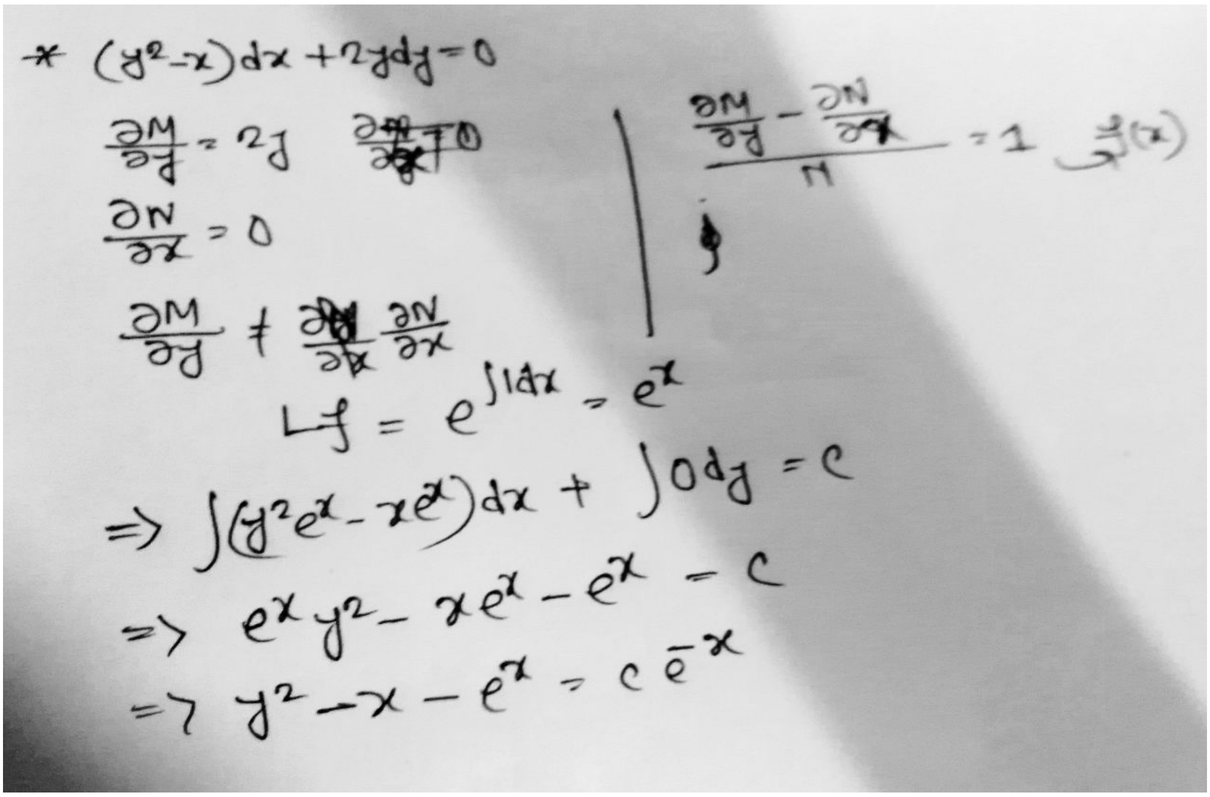
$\Rightarrow \dfrac{x^4}{4} - y^2\dfrac{x^2}{2} - \dfrac{x^2}{2}a - \dfrac{y^4}{4} - a\dfrac{y^2}{2} = C$

Reducible to Exact Differential Equation:

Steps:

Rule 1:

- Check the equation is exact or not, if not, go to next step
- $\frac{\frac{dM}{dy} - \frac{dN}{dx}}{N}$ and it is a function of only x or constant (not any single y) then go to next steps otherwise go to **Rule 2**
- Integrator factor IF = $e^{\int \frac{\frac{dM}{dy} - \frac{dN}{dx}}{N}}$
- Multiply the equation by the IF, then it is an exact equation now
- Follow the rules of EDE



* $(y^2-x)dx + 2y\,dy = 0$

$\frac{\partial M}{\partial y} = 2y$ $\quad \cancel{\frac{\partial N}{\partial x} \neq 0}$

$\frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = 1 \quad f(x)$

$Lf = e^{\int 1 dx} = e^x$

$\Rightarrow \int (y^2e^x - xe^x)dx + \int 0\,dy = c$

$\Rightarrow e^x y^2 - xe^x - e^x = c$

$\Rightarrow y^2 - x - e^x = c\,e^{-x}$

Correct Ans: $y^2 - x - 1 = ce^{-x}$

Rule 2:

- Check the equation is exact or not, if not, go to next step
- $\frac{\frac{dM}{dy} - \frac{dN}{dx}}{M}$ and it must be a function of only y or constant
- Integrator factor IF = $e^{-\int \frac{\frac{dM}{dy} - \frac{dN}{dx}}{M}}$ **[There’s a minus sign]**
- Multiply the equation by the IF, then it is an exact equation now
- Follow the rules of EDE

* $(y^2 dx) + (xy-1)dy = 0$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial y}{\partial x} = y$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy-1} \quad \boxed{\times}$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{2y-y}{y^2} = \frac{1}{y}.$ ✓

function of y

$IF = e^{-\int \frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}$

$\Rightarrow \int \frac{1}{y} y^2 dx + \int -\frac{1}{y} dy = c$

$\Rightarrow 0 - \ln y = c$

$\Rightarrow \ln y + c = 0$

Rules 1: **Homogenous Equation**

- $Mdx + Ndy = 0$ is a homogenous equation and $Mx + Ny \neq 0$ then,

  IF = $\frac{1}{Mx+Ny}$

  Multiply by the IF , then the equation is now an exact equation.

- $Mdx + Ndy = 0$ is a homogenous equation and $Mx + Ny = 0$ then,

  IF = $\frac{1}{Mx-Ny}$

  Multiply by the IF , then the equation is now an exact equation.

$$*\ (x^2y\,dx) - (x^3+y^3)\,dy = 0$$

$$\cancel{*\ x^2y\,dy - (x^3+y^3)\,dx = 0}$$

$$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$$

It is a homogenous functi
of order 3.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \ \boxed{x}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M\,\cancel{x^2y}} \ \boxed{x}$$

$$Mx + Ny = x^3y - x^3y + y^4 = -y^4 \neq 0$$

$$IF = \frac{1}{Mx+Ny} = -\frac{1}{y^4}$$

$$\Rightarrow \int \frac{-x^2}{y^3}\,dx + \int \frac{-y^3}{-y^4}\,dy = c$$

$$\rightarrow -\frac{x^3}{3y^3} + \ln y = c$$

<u>Initial Value Problem:</u>

<u>First Order Equation:</u>

$1)\frac{dy}{dx} = y$

$y(0) = 3$
$\Rightarrow \frac{dy}{y} = dx$
$or, \int \frac{dy}{y} = \int dx$
$or, lny = x + C$
$or, y = e^{x+c}$
$or, y = e^x.e^c$
$or, y = Ae^x$

$y(0) = 3$
$or, Ae^0 = 3$
$or, A = 3$

$y = 3e^x$

Linear Differential Equation with Constant Coefficient:

$$(\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}x}{dy^{n-1}} + P_2 \frac{d^{n-2}x}{dy^{n-2}} + ...P_n)y = Q$$

is a Linear Differential Equation if $P1, ..., Pn$ are the function of x or or constant.

If $P_1, ..., P_n$ are constant, then it is called **Linear Differential Equation with Constant Coefficient**.

If right hand side is zero or $Q = 0$, then it is called **Linear Homogenous Differential Equation.**

$D = \frac{d}{dx}$
$D^2 = \frac{d^2}{dx^2}$
.....
$D^n = \frac{d^n}{dx^n}$
$\Rightarrow (D^n + P_1 D_{n-1} + ... + P_n)y = Q$

1) If $Q = 0$,

$y = CF$

Where, CF = Complementary Function

<u>Complementary Function:</u>

- Auxiliary Equation (AE)

  $$(D^2 - 7D + 12)y = 0$$
  $$D^2 - 7D + 12 = 0...(i)$$

  Where (i) is an auxiliary equation

- Roots of AE

  $$D^2 - 7D + 12 = 0$$
  $$or, m^2 - 7m + 12 = 0$$
  $$m = 3, 4$$

- Nature of Roots

  $$Roots\ are\ real$$

<u>Rule 1:</u> Roots are **real and distinct**

$$y = C_1 e^{r_1 x} + ... + C_n e^{r_n x}$$

Where $r_1, ..., r_n$ are roots of the AE.

<u>Rule 2:</u> Roots are **real and repeated**

Suppose, roots are $r = -2, 2, 2$

$$y = C_1 e^{-2x} + (C_2 + C_3 x)e^{2x}$$

<u>Rule 3:</u> Roots are **imaginary**

$$r = \alpha \pm \beta i, ...$$

$$y = e^{\alpha x}(C_1 cos\beta x + C_2 sin\beta x) + ...$$

1) If $Q \neq 0$

$Q = Q(x)$, a function of x

$y = CF + PI$

Where, CF = Complementary Function

PI = Particular Integral

Particular Integral: **(NOT AS SIR DID)**

$$f(D).y = Q(x)$$

$$or, y = \frac{Q(x)}{f(D)}$$

Type 1: $Q(x) = e^{ax}$, $f(a) \neq 0$

- CF as Constant Coefficient
- $PI = \frac{e^{ax}}{f(a)}$
- $y = CF + PI$

Type 2: $Q(x) = e^{ax}$, $f(a) = 0$

- CF as Constant Coefficient
- $PI = x\frac{e^{ax}}{f'(a)}$
- $y = CF + PI$

Type 3: $Q(x) = sinax, cosax$

- CF
- $PI = \frac{sinax}{f(D)}$ Put $D^2 = -a^2$
- $D - Differentiation, \frac{1}{D} - Integration$
- $y = CF + PI$

Working Rules for Finding Particular Integral:

1. $\frac{1}{f(D)}x^m = [1 \pm F(D)]^{-1} x^m$.

2. $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ if $f(a) \neq 0$.

3. $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$; *or*, $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$ if $f(-a^2) \neq 0$.

4. $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$ *where, V is a function of x.*

CT:02

Bernoulli Differential Equation:

Definition: If P and Q are only functions of x or constants, then the differential equation of the form $\frac{dy}{dx} + Py = Qy^n$; $n \neq 0$ is called Bernoulli's equation

The Bernoulli Differentiation looks like,

$$\frac{dy}{dx} + Py = Qy^n$$

How to solve ? → Convert it to LDE. We have to vanish y^n .

Steps:

- Dividing the equation by y^n (besides Q)
- $z = y^m$ [besides P] and differentiate it w.r.t x
- Convert it to LDE

Problems:

- $\frac{dy}{dx} - \frac{1}{x}y = xy^2$

① $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{xy} = x$

$$\boxed{\begin{aligned} &y^{-1} = z \\ \Rightarrow\ &y^{-2}\frac{dy}{dx} = -\frac{dz}{dx} \end{aligned}}$$

$$-\frac{dz}{dx} - \frac{1}{x}z = x$$

$$\Rightarrow \boxed{\frac{dz}{dx} + \frac{1}{x}z = -x}$$ Converted into LDE

$$\frac{dy}{dx} = x^3 y^3 - x y$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3$$

$$\frac{1}{y^2} = z$$

$$\Rightarrow y^{-3}\frac{dy}{dx} = \frac{-dz}{2dx}$$

$$-\frac{dz}{2dx} + x \cdot z = x^3$$

$$\Rightarrow \frac{dz}{dx} - 2xz = x^3 \rightarrow \text{LDE}$$

UC Method:

UC Function:

A function $f(x)$ is **UC function**, if it is either

- x^n, where $n \geq 0$
- e^{ax}, where $a \neq 0$
- $sin(bx+c)$, $cos(bx+c)$, where $b \neq 0$
- any function that is a finite product of two or more functions of these three types

UC Set:

Definition: Given a UC function $f(x)$. We call UC set of $f(x)$, to the set of all UC functions consisting of $f(x)$ itself and all linearly independent functions of which the successive derivatives of $f(x)$ are either constant multiples or linear combinations.

	UC function	UC set
1	x^n	$\{x^n, x^{n-1}, x^{n-2}, \ldots, x, 1\}$
2	e^{ax}	$\{e^{ax}\}$
3	$\sin(bx + c)$ or $\cos(bx + c)$	$\{\sin(bx + c), \cos(bx + c)\}$
4	$x^n e^{ax}$	$\{x^n e^{ax}, x^{n-1}e^{ax}, x^{n-2}e^{ax}, \ldots, xe^{ax}, e^{ax}\}$
5	$x^n \sin(bx + c)$ or $x^n \cos(bx + c)$	$\{x^n \sin(bx + c), x^n \cos(bx + c), x^{n-1} \sin(bx + c), x^{n-1} \cos(bx + c), \ldots, x \sin(bx + c), x \cos(bx + c), \sin(bx + c), \cos(bx + c)\}$
6	$e^{ax} \sin(bx + c)$ or $e^{ax} \cos(bx + c)$	$\{e^{ax} \sin(bx + c), e^{ax} \cos(bx + c)\}$
7	$x^n e^{ax} \sin(bx + c)$ or $x^n e^{ax} \cos(bx + c)$	$\{x^n e^{ax} \sin(bx + c), x^n e^{ax} \cos(bx + c), x^{n-1}e^{ax} \sin(bx + c), x^{n-1}e^{ax} \cos(bx + c), \ldots, xe^{ax} \sin(bx + c), xe^{ax} \cos(bx + c), e^{ax} \sin(bx + c), e^{ax} \cos(bx + c)\}$

Problems:

1)

* $D^3 - D = 4e^{-x} + 3e^{2x}$

Soln.

Corresponding homogenous equation : $D^3 - D = 0$

Characteristic equation,

$$r^3 - r = 0$$
$$\Rightarrow r(r-1)(r+1) = 0$$
$$\Rightarrow r = 0, 1, -1$$

Fundamental set of the equation $F: \{1, e^x, e^{-x}\}$

Complementary solution :

$$y_c(x) = c_1 + c_2 e^x + c_3 e^{-x}$$

Non-homogenous term is: $b(x) = 4e^{-x} + 3e^{2x}$

It is the linear combination of e^{2x} and e^{-x}.

The UC set of e^{2x} is $S_1 = \{e^{2x}\}$

The UC set of e^{-x} is $S_2 = \{e^{-x}\}$

S_1 and S_2 are not equal nor ~~remains~~ one is included in the other.

Since, $S_2 = \{e^{-x}\}$ contains elements in $F = \{1, e^x, e^{-x}\}$ the we multiply the elements in S_2 by x, and we obtain $S_2^* = \{xe^{-x}\}$ that does not ~~a~~ contain element of F.

[Multiply till $S_2 \not\in F$ (any)]

Form a linear combination of the elements in S_1 and S_2^*, using unknown coefficients.

$$y_p(x) = Axe^{-x} + Be^{2x}$$

To determine the unknown coefficient, substitute the linear combination in the equation.

We must compute the first, second and third derivative,

$$y_p'(x) = A\{x(-e^{-x}) + e^{-x}\} + 2Be^{2x}$$

$$y_p''(x) = A\{x e^{-x} - 2e^{-x}\} + 4Be^{2x}$$

$$y_p'''(x) = A\{-xe^{-x} + e^{-x} + 2e^{-x}\} + 8Be^{2x}$$

$$= A\{-xe^{-x} \cancel{+e} + 3e^{-x}\} + 8Be^{2x}$$

Replacing into the equation,

$$A(-xe^{-x} + 3e^{-x}) + 8Be^{2x} - A(-xe^{-x} + e^{-x}) - 2Be^{2x} = 4e^{-x} + 3e^{2x}$$

Or, $$2Ae^{-x} + 6Be^{2x} = 4e^{-x} + 3e^{2x}$$

Then, $2A = 4 \Rightarrow A = 2$

$6B = 3 \Rightarrow B = 1/2$

Therefore, $y_p(x) = 2x e^{-x} + \frac{1}{2} e^{2x}$

The general solution of the non-homogenous equation is,

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow y(x) = c_1 + c_2 e^{x} + c_3 e^{-x} + 2xe^{-x} + \frac{1}{2} e^{2x}$$ (Ans)

(comparing/equating left-hand side to right –hand side… 2A = 4 ….)

2)

* $D^4 + 8D^2 + 16y = x^3 \sin 2x + x^2 \cos 2x$

Sol^n:

Corresponding homogenous equation: $D^4 + 8D^2 + 16y = 0$

Characteristics equation : $r^4 + 8r^2 + 16 = 0$

$\Rightarrow (r^2+4)(r^2+4) = 0$

$\Rightarrow r = \cancel{2i}, \pm 2i, \cancel{2i}, \pm 2i$

Fundamental set of the equation, $F = \{ \sin 2x, x\sin 2x, \cos 2x, x\cos 2x \}$

Complementary solution,

$$y_c = \cancel{c}\, e^{0x}\left[c_1\cos 2x + c_2 \sin 2x + x\, c_3 \cos 2x \cancel{+ c_3} \cancel{+ c_4 \sin 3x (bx)}\; x\, c_4 \sin 2x\right]$$

$$\cancel{= c_1\cos 2x + c_2 \cos \sin 2x + c_3 \sin 2x}$$

$$= c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 2x + c_4 \sin 2x$$

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Non-homogenous term : $x^3 \sin 2x + x^2 \cos 2x$

The UC set of $x^3 \sin 2x$, $S_1 = \{ x^3 \sin 2x, x^3 \cos 2x, x^2 \sin 2x, x^2 \cos 2x, x \sin 2x, x\cos 2x, \sin 2x, \cos 2x \}$

The UC set $x^2 \cos 2x$, $S_2 = \cancel{\{ x^2 \cos x, x^2 \sin x, x\cos x, x \sin x, \cos x, \sin x \}}$

$\{ x^2 \cos 2x, x^2 \sin 2x, x\cos 2x, x \sin 2x, \sin 2x, \cos 2x \}$

Since S_2 is included in S_1, we disregard S_2.

S_1 contains element of F, then we multiply S_1 by x^2,

$*S_1 = \{ x^5 \cos 2x, x^5 \sin 2x, x^4 \sin 2x, x^4 \cos 2x, x^3 \sin 2x, x^3 \cos 2x, x^2 \sin 2x, x^2 \cos 2x \}$

Form a linear combination of the elements in $*S_1$ using unknown coefficients.

The particular solution is,

$$y_p = Ax^5\sin 2x + Bx^5\cos 2x + Cx^4\sin 2x + Dx^4\cos 2x + Ex^3\sin 2x + Fx^3\cos 2x + Gx^2\sin 2x + Hx^2\cos 2x$$

Trajectories:

Trajectories: A curve which cuts every member of a given family of curves is called trajectories.

Orthogonal Trajectories: If a curve cuts every member of given family at **right angles** ($\alpha = 90$), then it is called Orthogonal Trajectory.

Working Rule:

- Differentiate the given equation of family of curve and **eliminate parameter/constant**
- Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$
- Solve this new DE and get the orthogonal trajectory

Orthogonal Trajectories :

* Find orthogonal trajectory of $y = ax^2$

$\Rightarrow$ Given that, $y = ax^2$

$$\Rightarrow \frac{dy}{dx} = 2ax = 2\cdot\frac{y}{x^2}x = 2\cdot\frac{y}{x} \quad [\text{step } 1]$$

$$\Rightarrow -\frac{dx}{dy} = 2\cdot\frac{y}{x} \quad [\text{step } 2]$$

$$\Rightarrow -x\,dx = 2y\,dy$$

$$\Rightarrow -\frac{x^2}{2} = 2\cdot\frac{y^2}{2} + b^2 \quad [\text{step } 3]$$

$$\Rightarrow \frac{x^2}{2b^2} + \frac{y^2}{b^2} = 1$$

Oblique Trajectory: A curve that intersects the curve of family at a **constant angle** $\alpha \neq 90$ is called **Oblique Trajectories**

Working Rule:

- Differentiate the given equation of family of curve and **eliminate parameter/constant,** denote it as $f(x, y)$
- $\frac{dy}{dx} = \frac{f(x,y)+tan\alpha}{1-f(x,y)tan\alpha}$ and solve the **DE**

Laplace Transform

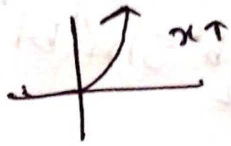
Definition: Let f(t) be a function of t defined for $0 \leq t < \infty$

Then, Laplace transform of f(t) denoted by $\mathscr{L}\{f(t)\}$ or $F(s)$, is defined by

$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{st} f(t)\,dt$

Reason why **s > 0 || s < 0 || (a-s) < 0** →

$$* \quad e^{\infty} = \infty$$



$$* \quad e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

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Problems:

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$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} \cdot 1 \, dt$$

$$= \lim_{N \to \infty} \left(\int_0^{N} e^{-st} \, dt \right)$$

$$= \lim_{N \to \infty} \left(\left[\frac{e^{-st}}{-s} \right]_0^N \right)$$

$\infty \times 0 = 0 \rightarrow \boxed{s > 0}$

$\infty \times \square \rightarrow \infty$

$$= \lim_{N \to \infty} \left(\frac{1}{-s} e^{-sN} - \frac{1}{-s} e^{-s(0)} \right)$$

$$= 0 + \frac{1}{s} = \frac{1}{s} \quad \boxed{s > 0}$$

* $\mathcal{L}\{t^n\}$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \cdot t^n \cdot dt$$

Let,

$$st = u \Rightarrow dt = \frac{du}{s}$$

t	0	∞
u	0	∞

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-u} \cdot \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} \cdot u^n \cdot du$$

$$= \frac{1}{s^{n+1}} \overline{|n+1}$$

$$\mathcal{L}\{t^n\} = \frac{\overline{|n+1}}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

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* $\mathcal{L}\{\cos bt\}$

$$\mathcal{L}\{\cos(bt)\} = \int_0^\infty e^{-st}\cos(bt)\,dt$$

D	I
$+\cos bt$	e^{-st}
$-b\sin bt$	$\frac{e^{-st}}{-s}$
$+b^2\cos bt$	$\frac{e^{-st}}{+s^2}$

Now, $$\int_0^\infty e^{-st}\cos(bt)\,dt = +\cos bt\cdot\frac{e^{-st}}{s} - \left(-\sin bt\cdot\frac{e^{-st}}{s^2}\right)b - b^2\int_0^\infty \cos bt\cdot\frac{e^{-st}}{s^2}\,dt$$

$$+\frac{b^2}{s^2}\int \cos bt\cdot e^{-st}\,dt \qquad +b^2\int \cos bt\cdot\frac{e^{-st}}{s^2}\,dt$$

$$\Rightarrow \frac{s^2+b^2}{s^2}\int e^{-st}\cos(bt)\,dt = +\cos(bt)\cdot\frac{e^{-st}}{s} + \sin(bt)\cdot\frac{e^{-st}}{s^2}\cdot b$$

$$\Rightarrow \int e^{-st}\cos(bt)\,dt = \frac{s^2}{s^2+b^2}\left[\cos(bt)\cdot\frac{e^{-st}}{s} - b\frac{e^{-st}}{s^2}\cdot\sin(bt)\right]$$

$$\mathcal{L}\{\cos(bt)\} = \lim_{N\to\infty}\left[\int_0^N e^{-st}\cos(bt)\,dt\right]$$

$$= \frac{s^2}{s^2+b^2}\times\frac{1}{s} \qquad [s>0]$$

Hence, $$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$$

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* $$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\cdot e^{at}\,dt$$

$$= \lim_{N\to\infty}\int_0^N e^{-t(s-a)}\,dt$$

$$= \lim_{N\to\infty}\left\{\left[\frac{e^{t(a-s)}}{a-s}\right]_0^N\right\} \qquad [a-s<0]$$

$$= -\frac{1}{a-s} \quad [\cancel{s>0}] \quad = \frac{1}{s-a}$$

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