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Discrete Mathematics

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Lecture from : Dr. Gajendra Purohit

Link : https://www.youtube.com/watch?v=wGLTV8MgLlA&list=PLU6SqdYcYsfJ27O0dvuMwafS3X8CecqUg&ab_channel=Dr.GajendraPurohit

Discrete : Distinct or not connected

Discrete mathematics is the subject of discrete objects. Such a finite set.

Set & It's type:

Set : A collection of well defined object.

Types:

- Empty Set : a set having no element
- Singleton Set: a set having only one element
- Subset : A set is called a subset of B if for all $x \epsilon A => x \epsilon B. A \subseteq B$
- Power Set : The family (set) of all subset of a set A , $P(A) = 2^n$

Operation:

- Union : $\{x | x \epsilon A or x \epsilon B\}$
- Intersection : $\{x | x \epsilon A and x \epsilon B\}$
- Difference : $\{x | x \epsilon A and x \not\epsilon B\}$
- symmetric difference (⊕) $\{x | x \epsilon A \ \ and \ \ x \ \not\epsilon B \ \ or \ \ x \epsilon B \ \ and \ \ x \ \not\epsilon A\}$

Cartesian Product: A X B = {(x1, y1), (x1, y2), (x2, y1), …. } where, x belongs to A and y belongs to B

…

[Some math will be added]

Principle of Mathematical Induction:

A proposition P and defined on the positive integer N. If:

- P(1) is true
- P(k+1) is true wherever when P(K) is true.

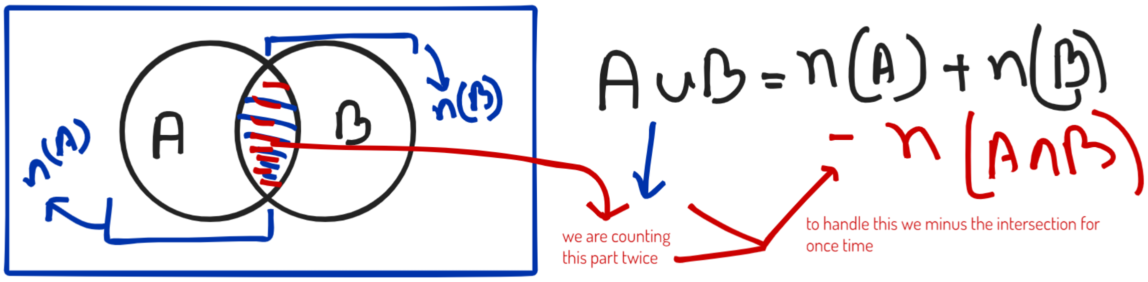
The, P is true for every position integer.

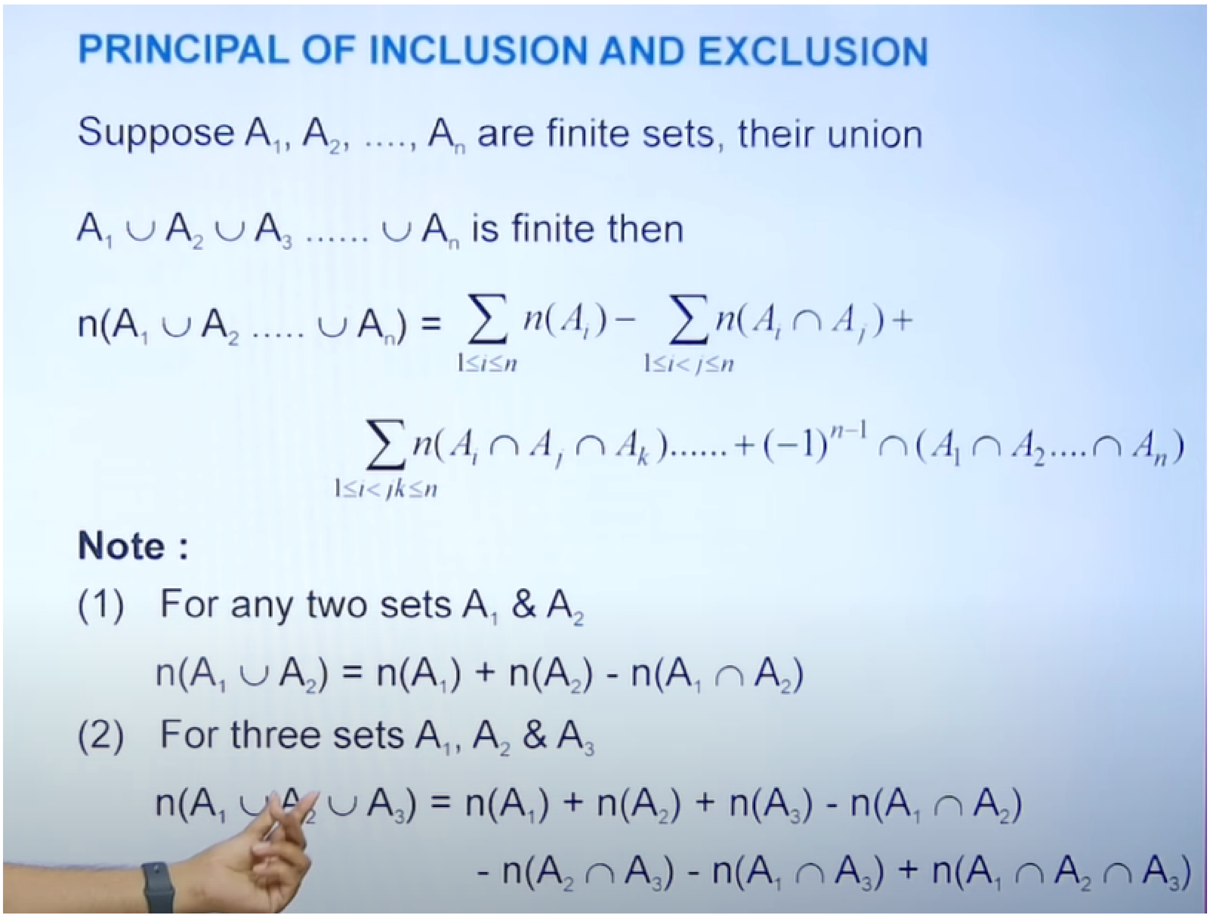
[some math]

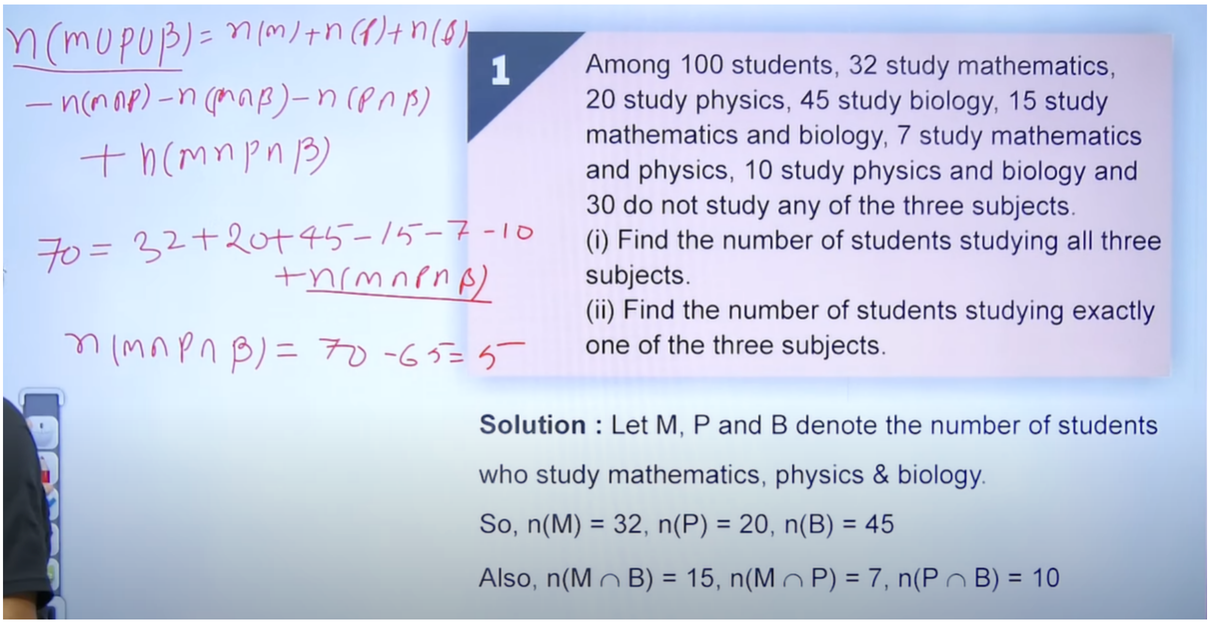
Venn Diagram:

Pictorial representation of a set by area written within circle with elements representation by point written within them is known as Venn Diagram.

Principle of Inclusion and Exclusion:







Relation:

Let A & B two sets and some elements of A be in certain correspondence with some elements of B then that correspondence defines a relation between elements of those two sets.

→ A - Domain

→ B - Co-domain

→ Correspondence value of A in B : Range

Properties of Relation:

1) Reflexive Relation: If A relation to R, if for all value x belongs to A and (x,x) belongs to R.

    - Irreflexive → not reflexive

2) Symmetric : (a, b) belongs to R and (b,a) belongs to R **(a≤b & b≤a i for all)**

3) Transitive : (x,y) belongs to R and (y,z) belongs to R and (x,z) belongs to R **(a≤b, b≤c, a≤c for all)**

Equivalence Relationship: If Relation R is

- Reflexive
- Symmetric
- Transitive

Example : **R = {(1,1), (1,2), (2,1), (2,2)}**

Note : → x divides y = y/x

→ x is divided/divisible by y = x/y

Composition of Relation:

(https://www.youtube.com/watch?v=e3HJN-grUrk)

* S o R → zeta aage oita sheshe
zeita sheshe oita aage (aage gele, baghe khay)
have to check one by one like below

R = {(2, 2), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (5, 3)}
S = {(2, 3), (2, 5), (3, 4), (3, 5), (4, 2), (4, 3), (4, 5), (5, 2), (5, 5)}.

S o R = (2,3) (2,4) (2,2) (3,3)

Partial Order Relation : (Out of Syllabus - probably)

- Reflexive
- Anti-Symmetric (a≤b & b≤a if only a==b)
- Transitive (a≤b, b≤c, a≤c for all)

The partial order relation R, is called **POSET.**

Total Order Relation: (Out of Syllabus - probably)

- Must be POSET
- Every elements must be connected to be all

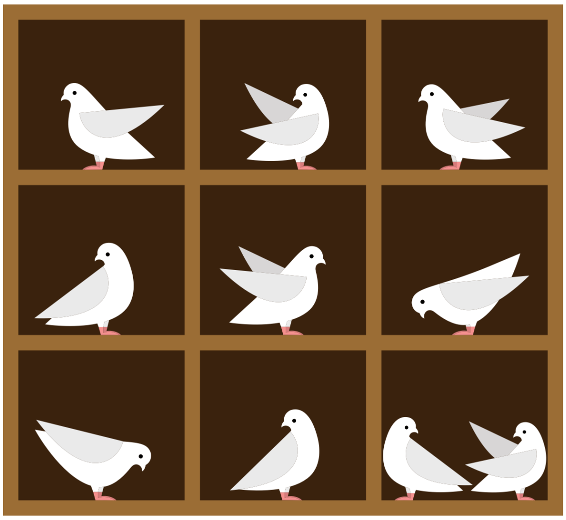
Is named as **ToSet.**

Note: aRb → a related to b

Hasse Diagram: (Out of Syllabus - probably)

- Partial order relation
- graph is upward

Pigeonhole Principle:



If (N+1) or more objects in N boxes then, there is at-least one box where there are two objects.

If n pigeonhole has kn+1 pigeons then at least one pigeonhole has k+1 (minimum) or more pigeons.

- How many teachers need if at least 4 teachers have same birth-month ?

  - $k + 1 = 4 \Rightarrow k=3$

    $n = 12$

    $kn+1 = 3*12 + 1 = 37$

Function :

- *If an input gives more than one output, than it's not a function*

Domain : All defined inputs

Range : All outputs

Co-domain : All possible outputs

Kinds of Function:

One-One Function:

- $f(x) \neq f(y)$, if $x \neq y$, [x,y belongs to Dom A]
- $f(x) = f(y)$, if $x=y$

Prove:

$f(x) = f(y)$

….

$\Rightarrow x=y$, (only one $\rightarrow$ one-one function, not more than 1 such that $x=y$ and $x = -y$ or something else)

!one-one function = many-one function

Onto Function:

- Range = Co-domain

Prove :

1) Find the range and compare it to co-domain

2) how to find range :

$y = f(x)$

….

$\Rightarrow$ x= something and compare it to range .. (It may be same constant number or just y+something like expression.. then y+something expression's data type needs to compare with the Range (R, N, Z etc.))

Bijective Function:

- Onto and one-one function at same time

Prove:

1) Prove, f(x) is one one

2) Prove, f(x) is onto

Problem:

Step 1 of 1

Given function is $f(x) = 2x + 1$

$f(x_1) = 2x_1 + 1$

and, $f(x_2) = 2x_2 + 1$

Let $f(x_1) = f(x_2)$

$2x_1 + 1 = 2x_2 + 1$

$2x_1 = 2x_2$

$x_1 = x_2$

If $f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

The function is one-to-one.

Step 2 of 2

Now we need to proceed with the second check :

Let $y = 2x + 1$

$x = \frac{y-1}{2}$

$\Rightarrow \frac{y-1}{2}$ = x .. . . equation (i)

From equation (i) we can say each value of $\frac{y-1}{2}$ gives a unique value of ' x '. Therefore the function is onto.

As this function passes both the checks so that it's bijective from R to R.

Final answer

The function $f(x)$ = 2x + 1 is a bijective function from R to R.

Note : If R → then negative values are acceptable too

Z+ → only positive values are acceptable

Find Domain :

- when the function will be defined

Find Range:

y = f(x)

⇒ x = … = f(y)

Find the domain for f(y).

Inverse Function:

- Must be bijective function

Find Inverse Function:

- First show that it's a bijective function (using two-three lines desc or full proof)
- y = f(x)
  ⇒ x = …
  ⇒ f(y) = …

Where f(y) is the inverse function of f(x)

Explanation

Given function is $f(x) = x^2$.
We have to calculate the value of $f^{-1}\{(1)\}$.

- First, calculate the inverse function. The process of to find f Let $y = f(x)$. Then calculate the value of x in term of y.
- Plug $y = 1$ in the inverse function.

Step 1 of 2

Given function is $f(x) = x^2$.
Let $f(x) = y$.
$\therefore y = x^2$
$x^2 = y$
$y = \pm\sqrt{x}$.
Therefore the inverse function of the function is $f^{-1}(y) = \pm\sqrt{y}$.

Step 2 of 2

Plug $y = 1$
$f^{-1}(1) = \pm\sqrt{1}$
$f^{-1}(1) = \pm 1$

Final answer

- Hence $f^{-1}(1) = \{1, -1\}$

Composite Function:

if f : A→B and g : B→C, gof : A→C then gof(x) is known as Composite function.

- f(x)….g(x) ⇒ gof(x) ⇒ g(f(x)) = x+…
- gof(x) is known as composite function.

Problems:

1)

Tip

In mathematics, an injective function (also known as injection, or one-to-one function) is a function that maps distinct elements of its domain to distinct elements of its codomain.[1] In other words, every element of the function's codomain is the image of *at most* one element of its domain.
The function f is one to one if and only if f(a)=f(b) implies that a=b for all a and b in the domain
In mathematics, function composition is an operation that takes two functions f and g, and produces a function h such that h(x) = f(g(x)) denoted by $(f \circ g)$ (x)

Explanation

Given g: $A \to B$ is one-to-one and f: $B \to C$ is one-to-one.
To prove : $f \circ g$ is one-to-one.
f is one-to-one : if $f(x) = f(y)$ then $x = y$.
g is one-to-one : if $g(x) = g(y)$ then $x = y$.
Now Let us assume $(f \circ g)(a) = (f \circ g)(b)$.
By using the definition of composition $f(g(a)) = f(g(b))$
Since f is one-to-one then $g(a) = g(b)$.
Since g is one-to-one then $a = b$
$\therefore f \circ g$ is a one-to-one function.

Final answer

By using the definition of one-to-one and composition ,we have shown that $f \circ g$ is a one-to-one function.

2)

Explanation

Given $f(x) = x^2 + 1$ and $g(x) = x + 2$
We need to calculate $f \circ g$ and g$\circ$ f

Step 1 of 2

For f(g(x)) here $g(x)$ will be an input to the function $f(x)$.
$(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5.$

Step 2 of 2

For $g(f(x))$ here $f(x)$ will be an input to the function $g(x)$.

$(g \circ f)(x) = g(f(x)) = g(x^2+1) = x^2 + 1 + 2 = x^2 + 3$

Final answer

Hence, By using the definition of composition we found that,
$(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5.$
and
$(g \circ f)(x) = g(f(x)) = g(x^2+1) = x^2 + 1 + 2 = x^2 + 3$.

3)

Tip

Functions with overlapping domains can be added, subtracted, multiplied and divided. If $f(x)$ and $g(x)$ are two functions, then for all x in the domain of both functions the sum, difference, product and quotient are defined as follows.

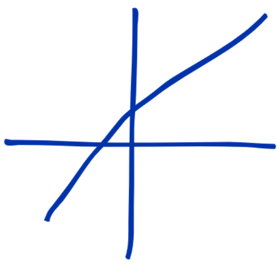
- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x) \times g(x)$
- $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

Explanation

Given $f(x) = x^2 + 1$ and $g(x) = x + 2$
So $(f+g)(x) = f(x) + g(x) = x^2 + 1 + x + 2 = x^2 + x + 3$.
and
$(fg)(x) = f(x) \times g(x) = (x^2+1) \times (x+2) = x^3 + 2x^2 + x + 2$.

Final answer

Hence, we have found $(f+g)(x) = x^2 + x + 3$ and $(fg)(x) = x^3 + 2x^2 + x + 2$.



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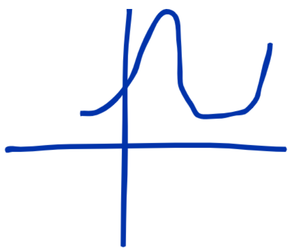

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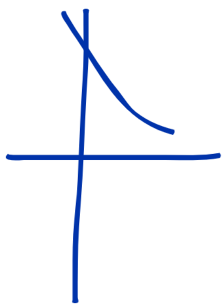


Bijective

Strictly Increasing : x < y ⇒ f(x) < f(y)

Strictly Decreasing : x < y ⇒ f(x) > f(y)

Floor and Ceil Function:

Rounding Functions						
	-3.5	-2.9	-1.1	1.1	2.9	3.5
FLOOR	-4	-3	-2	1	2	3
CEIL	-3	-2	-1	2	3	4
ROUND	-4	-3	-1	1	3	4
INT	-3	-2	-1	1	2	3
INT + SIGN	-4	-3	-2	2	3	4

Ceil : (+) → integer part + 1

(-) → integer part

Floor = (+) → integer part

(-) → integer part - 1

Recurrence:

- If $< a_n >$ is a sequence, an equation connecting a_n with a finite number of previous terms as $a_{n-1}, a_{n-2} ... a_{n-k}$ of the sequence is known as **Recurrence Relation.**

Example:

Fibonacci : 1 1 2 3 5 ..

where $a_n = a_{n-1} + a_{n-2}$, it's called **Recurrence Relation/Difference Equation.**

Order of Recurrence Relation:

- Difference between greatest suffix and least suffix

$a_n = a_{n-1} + a_{n-2}$

Order of Recur. Relation : $n - (n-2) = 2$

Linear Recurrence:

- Having no product of (more than 1) previous term

$\Rightarrow 0 = 5a_{n-1} + a_{n-2}$

Linear Recurrence with Constant Co-efficient :

$f(n) = c_0 a_{n-1} + ... + c_k a_{n-k}$

Sum:

$1 + 2 + 3 + 4 + ... + 5 = \frac{n(n+1)}{2}$

$x^0 + x^1 + ... + x^n = \frac{1-x^{n+1}}{1-x}$

Problems:

Check that $a_n = 2^n + 1$ is a solution to the recurrence relation $a_n = 2a_{n-1} - 1$ with $a_1 = 3$.

▾ Solution.

> First, it is easy to check the initial condition: a_1 should be $2^1 + 1$ according to our closed formula. Indeed, $2^1 + 1 = 3$, which is what we want. To check that our proposed solution satisfies the recurrence relation, try plugging it in.
>
> $$\begin{aligned} 2a_{n-1} - 1 &= 2(2^{n-1} + 1) - 1 \\ &= 2^n + 2 - 1 \\ &= 2^n + 1 \\ &= a_n. \end{aligned}$$
>
> That's what our recurrence relation says! We have a solution.

Example 2.4.3.

Solve the recurrence relation $a_n = a_{n-1} + n$ with initial term $a_0 = 4$.

▼ Solution.

To get a feel for the recurrence relation, write out the first few terms of the sequence: $4, 5, 7, 10, 14, 19, \ldots$. Look at the difference between terms. $a_1 - a_0 = 1$ and $a_2 - a_1 = 2$ and so on. The key thing here is that the difference between terms is n. We can write this explicitly: $a_n - a_{n-1} = n$. Of course, we could have arrived at this conclusion directly from the recurrence relation by subtracting a_{n-1} from both sides.

Now use this equation over and over again, changing n each time:

$$\begin{aligned} a_1 - a_0 &= 1 \\ a_2 - a_1 &= 2 \\ a_3 - a_2 &= 3 \\ \vdots \quad & \quad \vdots \\ a_n - a_{n-1} &= n. \end{aligned}$$

Add all these equations together. On the right-hand side, we get the sum $1 + 2 + 3 + \cdots + n$. We already know this can be simplified to $\frac{n(n+1)}{2}$. What happens on the left-hand side? We get

$$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}).$$

This sum telescopes. We are left with only the $-a_0$ from the first equation and the a_n from the last equation. Putting this all together we have $-a_0 + a_n = \frac{n(n+1)}{2}$ or $a_n = \frac{n(n+1)}{2} + a_0$. But we know that $a_0 = 4$. So the solution to the recurrence relation, subject to the initial condition is

$$a_n = \frac{n(n+1)}{2} + 4.$$

(Now that we know that, we should notice that the sequence is the result of adding 4 to each of the triangular numbers.)

Propositional Sentence: (OUT OF SYLLABUS)

An expression consisting of symbols, numbers or words, letters is called a propositional sentence if it is true or false.

- True Statement (T)

→ 2+3 = 5

- False Statement (F)

→ 2 + 3 = 1

Types of Propositional Sentences:

- Simple : Having one subject and one predicate

  → The flower is blue;

- Compound Preposition : Combination of more than one simple prepositional statement

  → The earth is round and revolves the sun.

Logical Operation

- And (^) / Conjunction

p	q	p^q
T	T	T
T	F	F
F	T	F
F	F	F

- Or(⌄) / Disjunction

  [⌄ → Ctrl + v]

p	q	p⌄q
T	F	T
T	T	T
F	T	T
F	F	F

- Negation(~) → Opposite

p	~p
T	F
F	T

Tautologies → if last column contain only T

Contradiction → if last column contain only F

Conditional Statement:

“if p then q”

Denoted by : → (implies that)

It works like a PROMISE.

Example *: If I would be the mayor (p), I will decrease the taxes(q)*

when p is false, q is ….. . . so it returns T.

p	q	p → q
T	T	T
F	T	T
T	F	F
F	F	T

Biconditional Statement:

“if p and only if q”

means : p→q and q → p = p ↔ q

denoted by : ↔

Example: *If I study hard, then I will pass.*

It also mean that, *If I pass, I studied hard.*

p	q	p ↔ q
T	T	T
F	T	F
T	F	F
F	F	T

Converse Statement : p → q, its converse statement q → p

Inverse of Conditional Statement : p → q, its inverse .. ~p → ~q

Contrapositive of Conditional Statement : p → q, its contrapositive , ~q → ~p

Some Important Law :

[Set theory formulas]

1) p → q = ~ p ⌄ q

Normal Form:

1) Disjunction Normal Form (DNF): (conjunction statement) disjunction (conjunction statement) …. **disjunction between conjunction(all)**

Example

→ **$(p \wedge q) \vee (r)$**

→ **$(p \wedge q) \vee (r \wedge p)$**

2) Conjunction Normal Form(CNF): conjunction between disjunction(all)

Example:

→ **$(p \vee q) \wedge (r \vee p)$**

[Math will be added]

Argument: (OUT OF SYLLABUS)

- Process by which a conclusion is obtained from given set of premises.
- Premises : Given group of proposition
- Conclusion: The proposition getting by given premises is called conclusion

Valid Argument : Let P1, … Pn are premises and Q is the conclusion, then the argument is valid iff **A(P1, …, Pn)** → **Q** is tautology.

Fallacy argument : If the argument is not valid

Note : iff = if and only if

[Some math will be added]

Finite State Machine - FSM Design → Skipped

Graph Theory:

- a mathematical structure of two set V (Vertices or Nodes) and E (edges) where V and E both are non-empty

Basic Terminology:

1) Trivial Graph: Consisting of 1 vertex only

2)Null Graph : consisting of n vertices and no edge

3) Directed Graph : Consist the direction of the edges

4) Undirected Graph: No direction of the edges (it works like bi-direction btw)

5) Self Loop : If a edge having the same vertex both of it's side **(self loop degree = 2, check number 11)**

6) Proper Edge : An edge which hasn't self loop

7) Multi edge : A collection of **≥2** edges having identically same start and end point

8) Simple Graph : Graph doesn't contain multi edge and self loop

8) Multigraph : not contain self loop but multi edge

9) Pseudo Graph: contain both self loop and multi edge

10) Incidence and Adjacency : An edge e joining two vertices V1 and V2, here e is called incident and V1, V2 are adjacent of e

11) Degree of Vertex : A vertex connects with **the number of edges**

12) Isolated Graph : Degree of a vertex is 0

13) Pendant Graph: Degree of a vertex is 1

14) Finite Graph: Finite numbers of vertices

15) Infinite Graph: Infinite numbers of vertices

Types of Graph:

16) Complete Graph: Every and each vertex connects with one another (E = v(v-1)/2)

17)Regular Graph: Every vertex has same degree

18) Bipartite Graph: Graph can be partitioned into two subset such a way that X has one end and Y has one end

19) Connected Graph : Undirected graph and every two vertices has a path means : **we can reach every vertex from any vertex a graph. [Every complete graph is connected graph]**

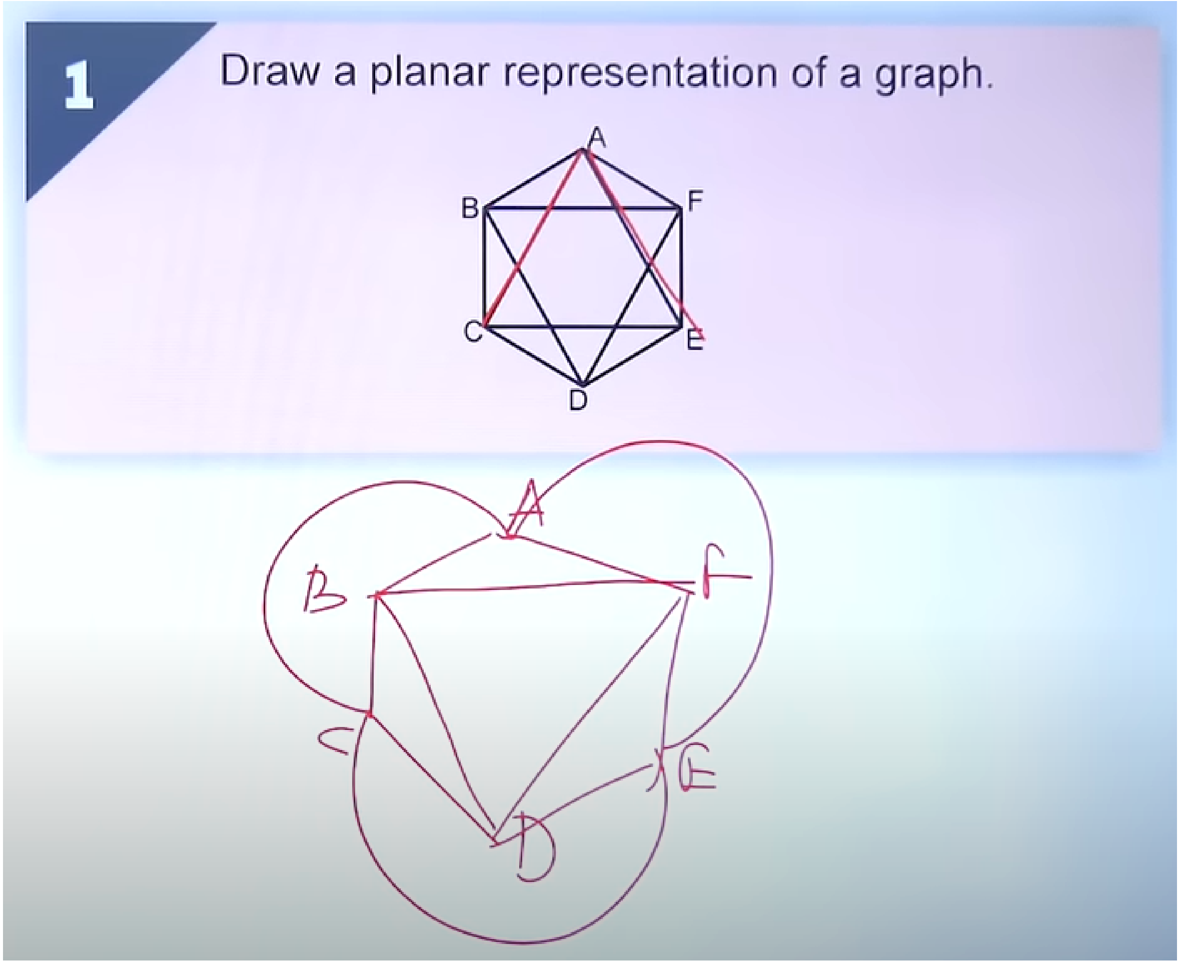
20) Complete Bipartite Graph : If every vertex in X is disjoint to every vertex in Y

21) Subgraph : Subset of V,E

22) Decomposition of Graph: Subgraph G1 and G2, G1 Union G2 = G ^ G1 Intersection G2 = NULL

23) Complement Graph: remove the edge and add edge where it doesn't in a graph

24) Planar Graph : if edges do not cross each other



Handshaking Theorem:

Vertex = 2 * Edges

Graph Representation :

1) Adjacent Matrix Representation :

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array}$$

$$A = \begin{bmatrix} & v_1 & v_2 & v_3 \\ v_1 & 1 & 1 & 0 \\ v_2 & 0 & 0 & 3 \\ v_3 & 1 & 0 & 0 \end{bmatrix}$$

if self loop then **A[i][i] = 2**

1) Incident Matrix Representation :

$$A = \begin{bmatrix} & e_1 & e_2 & e_3 \\ v_1 & 1 & 1 & 0 \\ v_2 & 0 & 1 & 1 \\ v_3 & 1 & 0 & 1 \end{bmatrix}$$

if self loop then **A[i][i] = 2**

3) Path Matrix Representation:

- Find the Adjacent matrix
- Path Matrix = $A + A^2 + ... + A^n$
    - n = number of nodes

Graph Isomorphism:

Two Graphs G1 and G2 will Isomorph iff,

- Vertices number are same
- Edges numbers are same
- Equal number of vertices with equal degrees
- Vertex correspondence and edge correspondence same

Homeomorphic Graph:

- If graph G1 and G2 can be obtained from same graph
- It could be isomorphic or not

Walk:

- Finite alternating sequence of vertex and edge, $v_1 e_1 v_3 e_4 v_5$
- Length of Wail : Number of edges visited
- Closed and open walk : if start and end vertex are same its close otherwise open
- Trail : Any walk having different edges
- Circuit : A closed trail
- Path: A walk which all vertices are not repeated
- Cycle: A closed path is cycle

Eulerian Graph:

- A path if it traverses each edge in the graph one and only once
- Edge can’t be repeated

- Edge < Vertex

Eulerian Circuit:

- A circuit if it traverses each edge in the graph one and only once
- Edges can’t be repeated
- Edge < Vertex

Hamiltonian Path:

- A path which contain **every vertex** of a graph
- Vertex can’t be repeated
- Edge > Vertex

Hamiltonian Circuit:

- A circuit passing through **all vertex** of a graph
- Vertex can’t be repeated
- Edge > Vertex

Hamiltonian Graph:

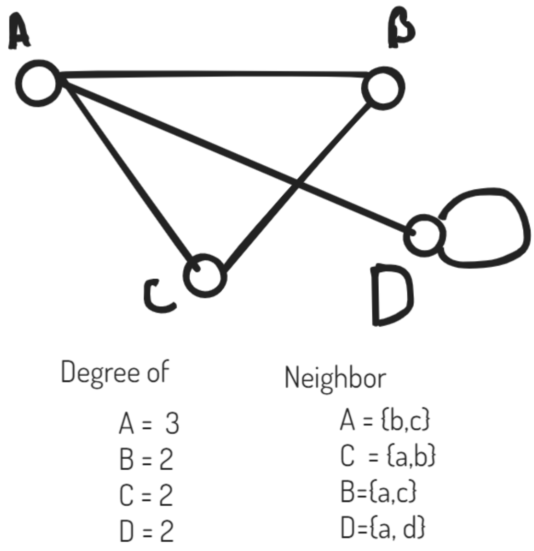
- A **connected** graph with a Hamiltonian circuit
- Vertex can’t be repeated
- Edge > Vertex

Weighted Graph:

- If a non-negative integer W (e) associated to each edge and this W (e) associate to corresponding edge

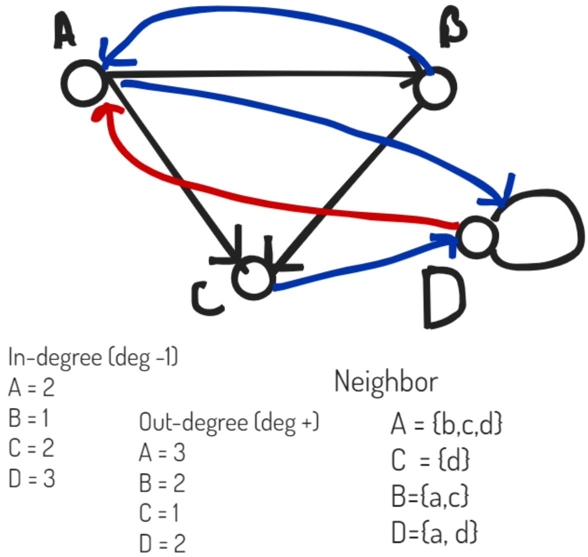
Degree and Neighbor

- **In Undirected Graph**
  - Degree of Node : total number of edges connected to the node
  - Neighbors : the nodes connected directly with the node



Correction: Degree of d = 3, cause self-loop : 2

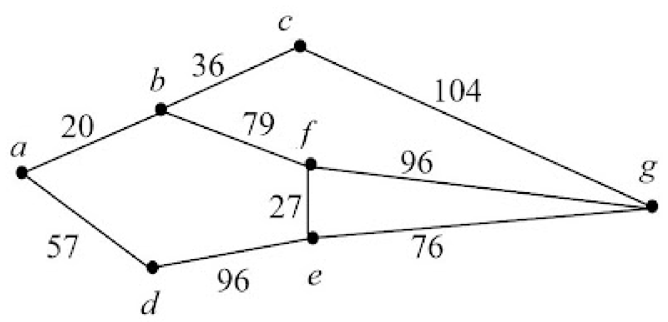
- **In Directed Graph**
  - In-degree : edges entering to the nodes
  - Out-degree: edges going out of the node
  - Neighbors : connected by out-degree nodes



A self-loop adds 2 to the degree of a vertex, or 1 to both indegree and outdegree in case of directed graphs

Dijkstra Algorithm: (Out of Syllabus)

- Find the shortest path of weighted graph



Soln Here
Finding shortest path from a to g using Dijkstra's Algorithm

V	a	b	c	d	e	f	g
a	$\boxed{0}$	∞	∞	∞	∞	∞	∞
b		$\boxed{20}$	∞	57	∞	∞	∞
c			$\boxed{56}$	57	∞	99	∞
d				$\boxed{57}$	∞	99	160
f					153	$\boxed{99}$	160
e					$\boxed{126}$		160
g							$\boxed{160}$

Shortest Path : a ⟵ b ⟵ c ⟵ g

REST : 23, 24, 25

Trees

- connected undirected graph with no simple circuit
- Rooted tree: particular node use as tree and

Combinatorics

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = {}^{n}C_{r}$$

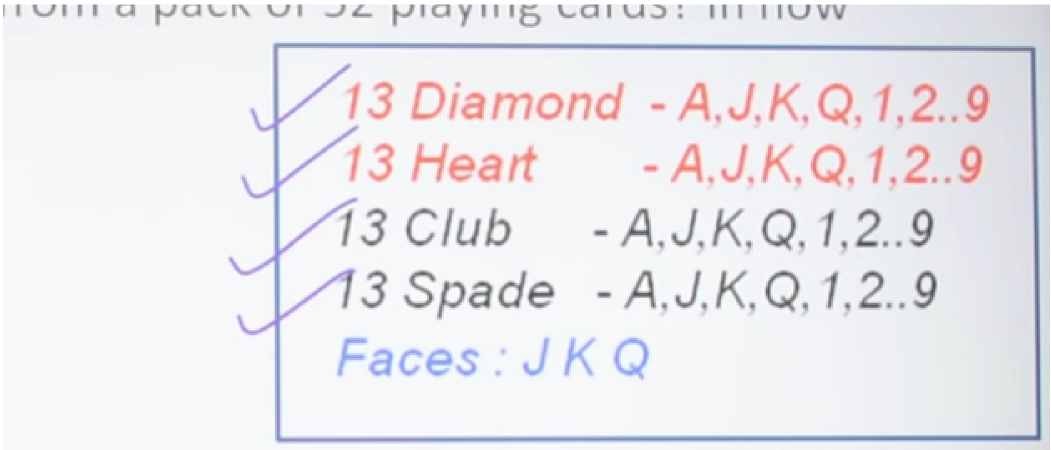
$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Binomial Theorem $(ax+by)^n$

Coefficient of $x^i y^j = \binom{n}{j}(a)^i(b)^j$

The keywords can help you get the answer easily:

The keywords like-selection, choose, pick, and combination-indicates that it is a combination question.
The Keywords like-arrangement, ordered, unique- indicates that it is a permutation question.



Product Rule : Subtask to solve a problem/and

Sum Rule : when these are different way of solving the same problems

<u>Number of bit strings of length four do not have two consecutive 1s</u>